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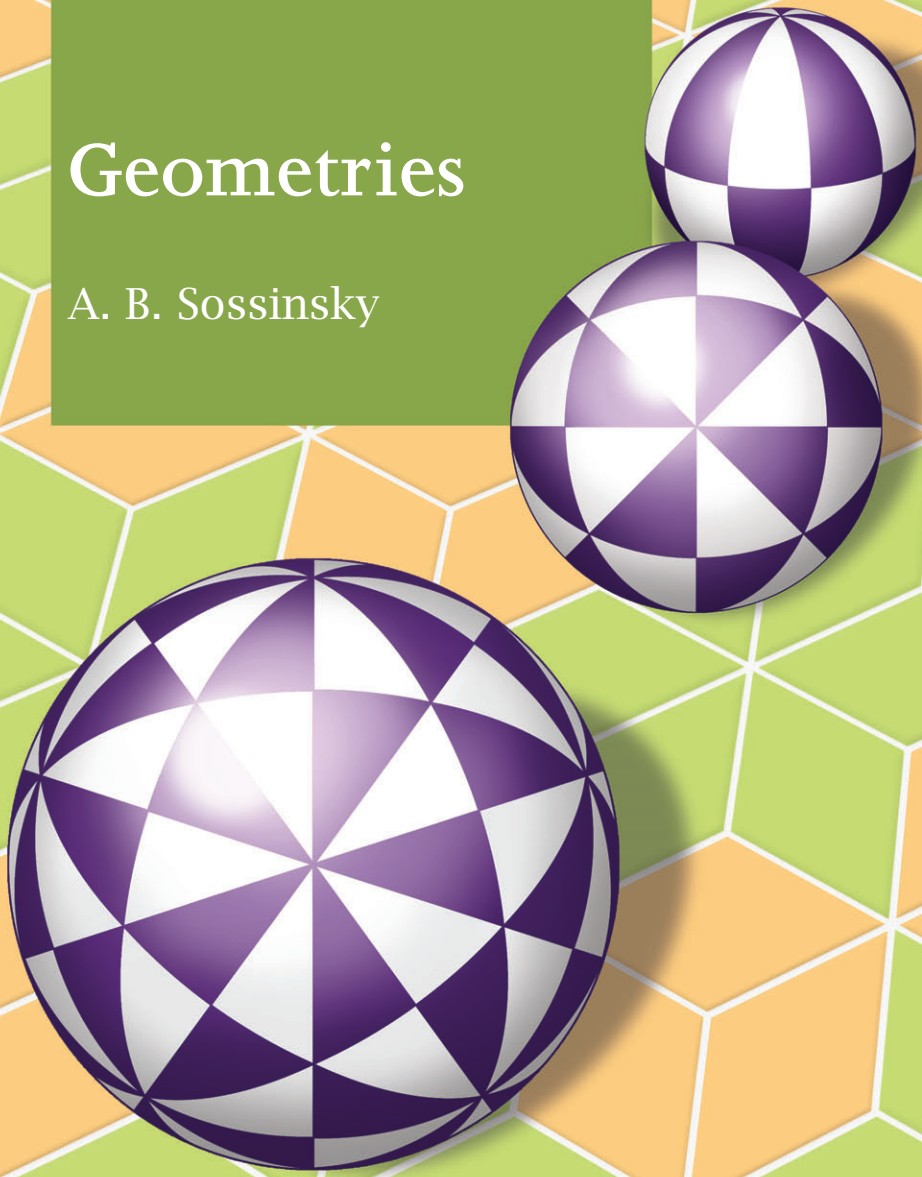
Geometries

A. B. Sossinsky



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Contents

Preface

This book is dedicated to the proposition that all geometries are created equal. This was first pointed out by Felix Klein, who declared that *each individual geometry is a set with a transformation group acting on it.* Here we shall study geometries from this point of view not only as individual objects, but also in their social life, i.e., in their relationships (called *morphisms* or *equivariant maps*) within their socium: the *category of geometries.*

Of course, some geometries are more equal than others. Accordingly, we will ignore the most common ones (affine and Euclidean geometries, vector spaces), assuming that they are known to the reader, and concentrate on the most distinguished and beautiful ones. (We assume that our readers are familiar with elementary Euclidean geometry; those who aren't may refer to Chapter 0, which is a précis of the subject, whenever the need arises.)

The reader should not be deceived by the words *groups, morphisms, categories* into thinking that this is a formal algebraic or (heaven forbid) an analytic (coordinate) treatment of geometric topics; it suffices to glance at the numerous figures in the book to realize that we constantly privilege the *visual aspect.*

Category *theory* is not used in this book, but we do use some basic category *language*, which, as the reader will see, is extremely natural

in the geometric context. Thus, Cayley's famous phrase: "projective geometry is all geometry" can be given a precise mathematical meaning by using the term *subgeometry* (which means "image by an injective equivariant map"). In the context of this book, it may be rephrased as follows: "The geometries studied in this book (including the three classical ones – hyperbolic, elliptic, and Euclidean) are (almost all) subgeometries of projective geometry."

There is very little in the main body of this book about the axiomatic approach to geometry. This is one of the author's biases: I believe that the classical axiom systems for, say, Euclidean and hyperbolic geometry are hopelessly outdated and no longer belong in contemporary mathematics. Their place is in the *history of mathematics* and in the philosophy of science. Accordingly, here they only appear in one chapter, devoted to the fascinating history of the creation of non-Euclidean geometry, while a detailed treatment of the axiom systems of Euclid and Hilbert is relegated to Appendices A and B.

The use of the plural (*Geometries*) in the title of the book indicates that, to my mind, there is no such *subject* as "geometry", but there are some concrete *mathematical objects* called geometries. In the singular, the word "geometry" should be understood as *a way of thinking about mathematics*, in fact the original one: in Ancient Greece, the word "geometry" was used as a synonym for "mathematics". One can and should think geometrically not only when working with circles and triangles, but also when using commutative diagrams, morphisms, or groups. The famous phrase written above the entrance to Plato's Academy

Let no one enter who is not a geometer

should also be displayed on the gates leading to the world of mathematics.

* * *

I will not give a systematic summary of the contents of this course in this Preface, referring the reader to the Table of Contents. Looking at it, the well-prepared reader may wonder why some of her/his

favorite geometric topics do not appear in this book among the "distinguished and beautiful" ones promised above. Let me comment on some missing topics, explaining why they are not treated here.

First, there is no *algebraic geometry* in this book. This is because the author believes that this beautiful field of mathematics belongs to algebra, not geometry. Indeed, the mathematicians doing algebraic geometry are typically algebraists, and this is not only true of the great French school (following Grothendieck and his schemes), but also of the more classical Russian school.

Neither is there any *differential geometry*: in its classical low-dimensional aspect it is usually developed in calculus books (where it indeed belongs); in its higher-dimensional modern aspect it is a part of analysis (under the title "Calculus on Manifolds") and topology (under the title "Differential Topology").

Other missing topics include *convex geometry* (part of analysis and more specifically optimization theory as "convex analysis"), *symplectic geometry* (part of classical mechanics and dynamical systems), *contact geometry* (part of differential equations), etc.

Of course, contact geometry (say) is formally a geometry in the sense of Klein. In fact, the ideology of transformation groups comes from Sophus Lie as much as (if not more than) from Felix Klein, but the context of Lie's beautiful contact geometry is definitely differential equations.

* * *

This book is based on lectures given in the framework of semester courses taught in Russian at the Independent University of Moscow to first-year students in 2003 and 2006, for which I prepared handouts, written in "simple English" and posted on the IUM web site. These lecture notes were published as a 100-page booklet by the Moscow Center of Continuous Mathematical Education in 2006 and used in geometry courses taught to Math in Moscow students of the Independent University.

The brevity of a short semester course (13 lectures) made me restrict the study of such classical geometries as hyperbolic and projective to dimension two, regretfully shelving the three-dimensional case. But then there are many occasions in the course for developing one's *intuition of space*, and indeed the general case is easier to treat from the linear algebra coordinate point of view than from the rather visual synthetic approach characterizing this course. For the reader who wants to go further, I strongly recommend the book by Marcel Berger [2]. I should add that, although my approach to the subject is very different from Berger's, I am heavily indebted to that remarkable book in several specific parts of the exposition. For those who would like to learn more about the axiomatic approach to the classical geometries, there is no better book, to my mind, than N.V. Efimov's *Higher Geometry* [6].

An important, if not the most important, aspect of this book are the problems, which appear at the end of each chapter. It is by solving these problems, much more than by learning the theory, that the reader will become capable of thinking and working geometrically. The sources of the problems are varied. Many were "stolen" from books written by my friend and favorite co-author Victor Prasolov. In many cases, I simply don't know where they originally come from. As handouts for the exercise classes, they were grouped together by Irina Paramonova, who contributed several, as did the other instructors conducting the exercise classes (Vladimir Ivanov and Oleg Karpenkov). I am grateful to all of the people mentioned above, and also to Mikhail Panov, Anton Ponkrashov, and Victor Shuvalov, who produced the computer versions of most of the illustrations, to M.I. Bykova, who corrected many errors in the original handouts, to Victor Prasolov, who found many more in the first draft of this book, and to the anonymous referees, whose constructive criticism was very helpful. Finally, I am indebted to Sergei Gelfand, without whose encouragement this book would never have been written.

Chapter 0

About Euclidean Geometry

The first chapter of this book is Chapter 1 (see p. 33 below); the present chapter (numbered 0) contains some standard facts about Euclidean geometry that can be regarded as prerequisites for reading this book. The author feels that the majority of readers should omit this chapter, even if they did not have a sound Euclidean geometry course in high school or college, and go right on to Chapter 1. Some concrete facts from Euclidean geometry are briefly recalled in the main text of the book when they are needed for our further exposition, and if they turn out to be unfamiliar or not too well remembered, then, by returning to Chapter 0, the reader can brush up on them, replacing these facts in the context of a systematic exposition of elementary Euclidean geometry.

Our exposition of Euclidean plane geometry is axiomatic. The axioms are chosen on the basis of Klein's approach to geometry, so that transformations play the key role, with a distance function specified from the outset, as is done in Kolmogorov's high school geometry textbooks. Our approach presupposes the knowledge of the *real numbers* and their main properties, some familiarity with the *language* of naive set theory (no serious knowledge of set theory *itself* is assumed),

and some experience in the kind of logical reasoning used in mathematics (e.g. arguing by contradiction or by induction, splitting a proof into various cases, and so on).

0.1. The axioms of Euclidean plane geometry

There are three types of undefined notions in our theory: *points*, (*straight*) *lines*, and the (*Euclidean*) *plane*. Points are denoted by various capital letters ($A, B, C, \ldots, P, Q, \ldots$, sometimes supplied with subscripts or superscripts), lines are denoted by small italics ($l, m, n, \ldots$, also possibly endowed with subscripts or superscripts), and the plane is denoted by $\mathbb{E}^2$. We are also given a *distance function* d, which assigns a nonnegative real number to each pair of points,

$$d : (A, B) \mapsto d(AB) \in \mathbb{R}_+.$$

In what follows, we will also use the more traditional notation $|AB|$ for the value of the distance $d(A, B)$ between two points.

These objects satisfy the following eight axioms.

I. *The plane $\mathbb{E}^2$ is the set of all points.*

II. *The family $\mathcal{L}$ of all lines consists of nonempty subsets of the plane $\mathbb{E}^2$.*

III. *For any two distinct points A and B there exists one and only one line $l = AB \in \mathcal{L}$ containing these two points.*

IV. *For any line $l \in \mathcal{L}$ there exists at least one point $P \in \mathbb{E}^2$ not contained in that line.*

To state the next axiom, we need a definition. Two lines l_1 and l_2 are called *parallel* if they coincide or if they have no common points; if the lines l_1 and l_2 are parallel, we write $l_1 \parallel l_2$.

V. *For any point $P \in \mathbb{E}^2$ and any line $l \in \mathcal{L}$ there exists one and only one line $l' \in \mathcal{L}$ parallel to l and containing P.*

VI. *The distance function d possesses the following properties:*

(i) *$d(A, B) = 0$ if and only if $A = B$;*

(ii) *$d(A, B) = d(B, A)$ for all $A, B \in \mathbb{E}^2$;*

(iii) $d(A,B)+d(B,C)\geq d(A,C)$ *for all* $A,B,C\in\mathbb{E}^2$; *the points* A,B,C *lie on the same line if and only if the previous inequality is in fact an equality;*

(iv) *for any point* P *on a line* $l\in\mathcal{L}$ *and any positive number* ρ *there are exactly two points* A *and* B *such that* $d(A,P)=d(P,B)=\rho$.

A few more definitions are needed to state the last axioms.

The first is that of isometry, which plays the key role in Klein's approach to the study of Euclidean geometry and of many other geometries (see Section 1.4 in Chapter 1). An *isometry* is a bijection β of $\mathbb{E}^2$ (i.e., a one-to-one transformation of $\mathbb{E}^2$ onto itself) that preserves distances:

$$d(\beta(P),\beta(Q))=d(P,Q)\quad\text{for all}\quad A,B\in\mathbb{E}^2.$$

Following the traditional terminology, we will often call subsets of the plane $\mathbb{E}^2$ (*geometric*) *figures*. Two figures $\mathcal{F}_1,\mathcal{F}_2\subset\mathbb{E}^2$ are called *congruent* if there exists an isometry φ of the plane that takes one onto the other, $\varphi(\mathcal{F}_1)=\mathcal{F}_2$.

The next definition is that of the "between" relation: if for three distinct points A,B,C, we have $d(A,B)+d(B,C)=d(A,C)$, then we say that the point B *lies between* the points A and C on the line $l=AC$. (Note that by axiom III we have $l=AC=CA=AB=BA=BC=CB$.) The set of all points lying between two distinct points A and B is denoted by (A,B) and said to be the *open interval* with *endpoints* A and B. If we add the endpoints to the points of an open interval (A,B), we obtain the *closed interval* (or *segment*) with *endpoints* A and B, which is denoted by $[A,B]$. Two distinct points O and A determine the *ray* with *origin* O passing through A, denoted by $[O,A\rangle$, as the union of the segment $[O,A]$ and all points B such that A lies between O and B.

Remark. In some expositions of Euclidean geometry (e.g., Hilbert's, see Appendix B), the "between" relation is an undefined relation. In most elementary textbooks on plane geometry, nothing is said about this relation itself, the exposition leans heavily on the illustrations, and a point is declared to lie between two other points provided it appears that way on the corresponding figure.

The union of two rays $[O, A\rangle$ and $[O, B\rangle$ with common origin O is called an *angle* and denoted $\angle AOB$ or $\angle BOA$; the point O is called the *vertex* of the angle. Two intersecting lines $AA' \cap BB' = O$, where O lies between A and A', and also between B and B', determine four angles: $\angle AOB, \angle A'OB', \angle AOB', \angle A'OB$; the first two, as well as the last two, are called *vertical* to each other. Two angles with a common ray located between their two other rays are called *adjacent*. If we remove the common ray of two adjacent angles, we obtain an angle that we call the (*geometric*) *sum* of the two given angles.

Two intersecting lines that form four congruent angles are called *perpendicular*, and these four angles are said to be *right angles.* If the two rays of a given angle $\angle AOB$ lie on one line and have no common points other than the vertex, we say that $\angle AOB$ forms *two right angles.*

We are now ready to formulate the next to last axiom.

VII. *For any line $l \in \mathcal{L}$ and any point $P \in \mathbb{E}^2$, there exists a unique perpendicular to l containing P.*

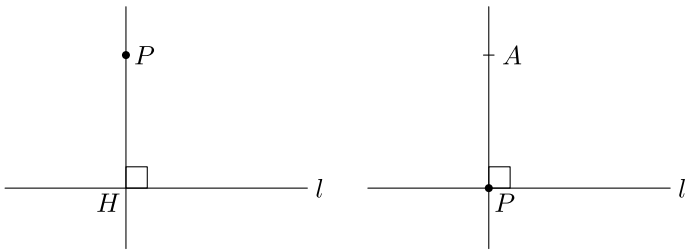


Figure 0.1. Dropping and raising perpendiculars.

Note that the axiom does not specify whether P lies on l or not, so that accordingly two different pictures illustrate this axiom (Figure 0.1). In the traditional teaching of geometry, one often uses the expression *raise the perpendicular to l from P* in the second case and *drop the perpendicular from P to l* in the first one. In both cases the intersection point of the perpendicular with l is often called the *foot* of the perpendicular.

One more definition, a very important one, is needed to formulate the last axiom. Let $l \in \mathcal{L}$ be a line; then the *reflection* in that line is the assignment $S_l : \mathbb{E}^2 \to \mathbb{E}^2$ that leaves each point of l fixed and takes any point $P \notin l$ to the point $P' := S_l(P)$ lying on the perpendicular drawn from P to l and such that $|PH| = |HP'|$, where H is the foot of the perpendicular and lies between P and P'.

VIII. *The reflection in any line is an isometry.*

If l is any given line and P is a point not on l, then the *half-plane* determined by l and P is the set of all points M of the plane such that the segment $[P, M]$ does not intersect l. It follows immediately from the definition of reflection (and the axioms) that *any reflection in l interchanges the two half-planes determined by l.*

0.2. Commentary

0.2.1. The axioms listed above, except for axiom V (the axiom of parallels) are intuitively obvious. In the standard school model of elementary school geometry, in which the plane (or rather part of it) is a piece of paper lying on a flat table, points are marks on that paper made by a well-sharpened pencil, lines are obtained by sliding the pencil along the edge of a ruler, and distances between points are measured in the usual way by means of that ruler, axioms I–IV and VI–VIII may actually be regarded as experimental facts. This is not true of the axiom of parallels (lines being infinitely long, we cannot extend them "to infinity" to verify experimentally how many parallels there are, if any). As to the reflection axiom (VIII), it can be modeled by placing a piece of tracing paper with a line drawn on it on our "paper plane" (on which a line l is also drawn) so that the two lines coincide, and then flipping the tracing paper around and placing it back on the paper plane so that the lines coincide as they did before; if a point P on the paper plane is given and its position on the tracing paper is denoted by P_1, the position of the point P_1 after the tracing paper is turned over will play the role of the image P' of P under the reflection; the fact that distances are preserved seems obvious (one cannot stretch or shrink the tracing paper).

0.2.2. Independence. The axioms I–VIII are *not* independent: some of their assertions can be derived from the other axioms. Moreover, the undefined notion of "straight line" can actually be rigorously defined by using axiom VI, the notion of set, and the undefined notion of "point". If that is done, axiom III becomes a theorem.

Hilbert constructed a rigorous axiom system for Euclidean geometry (see Appendix B) in which the axioms are independent (in the sense that there is no axiom that can be logically deduced from all the others). His approach is conceptually important for the foundations of mathematics, but is not very satisfactory from the pedagogical point of view.

0.2.3. Consistency. The axiom system above is consistent (i.e., no contradiction can be obtained from its axioms by correct logical arguments) provided that the theory of real numbers is consistent. This can be proved by constructing a *model* for this axiom system, i.e., supplying the undefined terms with concrete meanings within the theory of real numbers so that the axioms are theorems in that theory. This is done as follows: the undefined notion of "plane" is interpreted as $\mathbb{R}^2$, the set of ordered pairs of real numbers, "points" are pairs $(x, y) \in \mathbb{R}^2$, "lines" are sets of pairs satisfying linear equations $\alpha x + \beta y + \gamma = 0$, and the distance function is defined by the formula

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The fact that these interpretations of the undefined notions satisfy axioms I–VIII can be verified in a straightforward way.

0.2.4. Categoricity. The axiom system above is categorical, i.e., any two of its models are isomorphic in a certain natural sense that we do not specify here. Nor do we comment on the fairly straightforward proof of this fact.

0.2.5. Why the axiomatic approach? The reader may wonder why Euclidean plane geometry was described here by means of axioms, when there is a much simpler way of introducing it, namely by defining $\mathbb{E}^2 := \mathbb{R}^2$ as explained in Subsection 0.2.3. There are two reasons for this. The first is to give a tribute to the traditional teaching

of geometry (to some version of which the reader was originally subjected); the second is that the coordinate approach to geometry, to the author's mind, is an ugly caricature of what Euclidean geometry really is.

We must also warn the reader that our exposition is probably more rigorous and formalized than traditional ones in high school geometry textbooks, but also less detailed. In particular, here the proofs are either only sketched or omitted altogether (but the theorems appear in a logical order, so that each easily follows from the previous theorems and the axioms). Also there are no exercises in this chapter.

0.3. Rotations

0.3.1. Properties of isometries. Any isometry:

(i) *takes lines to lines;*

(ii) *takes open intervals, segments, rays to open intervals, segments, rays, respectively;*

(iii) *takes parallel lines to parallel lines;*

(iv) *takes perpendiculars to perpendiculars;*

(v) *takes angles to angles;*

(vi) *the composition of two isometries is an isometry.*

All these properties immediately follow from the corresponding definitions (and the axioms).

0.3.2. Definition of rotations. An *oriented angle* is an ordered pair of rays with common origin O (notation $\angle([O,A\rangle,[O,B\rangle)$, the point O is the *vertex* of the oriented angle). Let $\alpha := \angle([O,A\rangle,[O,B\rangle)$ be an oriented angle, let l and m be the lines OA and OB, respectively; assume that $l \neq m$; then the composition of reflections in the lines l and m, S_l and S_m, performed in that order, will be called the *rotation* with center O by the angle 2α and denoted by $R_{2\alpha}$. (The integer 2 in the last expression is not a misprint, a glance at Figure 0.2 shows where it comes from.)

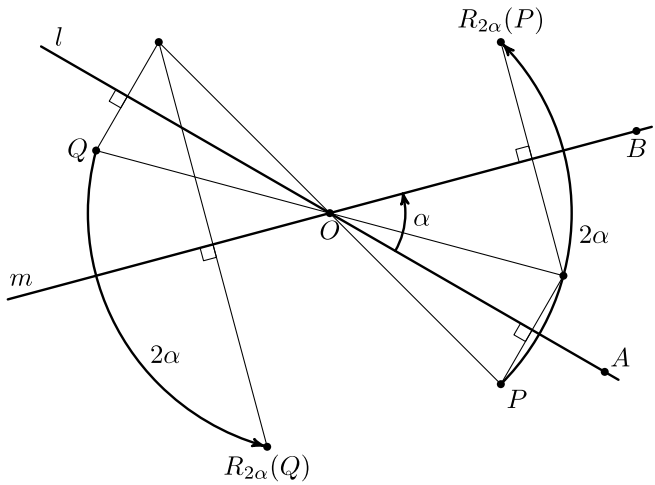


Figure 0.2. Rotation by 2α.

In the particular case in which the rotation is defined by an oriented angle $\alpha = \angle([O, A\rangle, [O, B\rangle)$ such that the lines OA and OB are perpendicular, the rotation with center O by the angle 2α will be called the (*central*) *symmetry* with *center* O and denoted by S_O.

A *circle* of *center* O and *radius* $r > 0$ is the set of all points M such that $|OM| = r$. If we are given a rotation by a certain angle $\alpha = \angle([O, A\rangle, [O, B\rangle)$ and some point $P \neq O$, its image P' under the rotation $R_{2\alpha}$ obviously lies on the circle of center O and radius $|OP|$, which justifies the term "rotation" (look at Figure 0.2 again). Similarly, and just as obviously, the image of some point $Q \neq O$ under the symmetry S_O with center O can be constructed by choosing the point Q' on the line OQ so that $|QO| = |OQ'|$ and O lies between Q and Q', which again justifies the terminology.

Proposition 0.3.3. *Rotations and central symmetries, being compositions of isometries, are isometries, and therefore possess properties* (i)–(v).

Theorem 0.3.4. *Vertical angles are congruent.*

This immediately follows by performing a symmetry centered at the vertex of the angle.

If we are given two parallel but not coinciding lines and a line cutting them both, then these three lines determine eight angles (see Figure 0.3) that can be split into two quadruples in each of which all four angles are congruent.

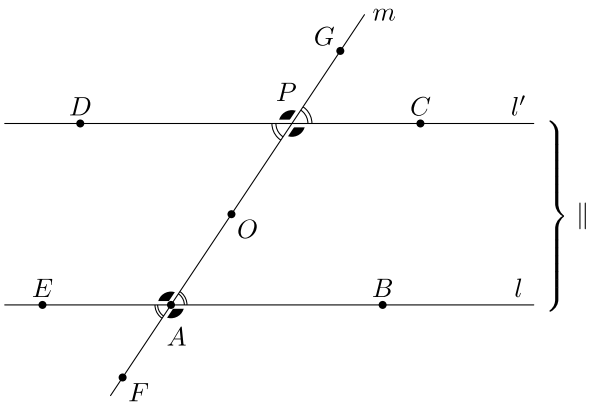


Figure 0.3. Congruent angles formed by a line cutting two parallels.

Let us formally introduce the notation appearing in Figure 0.3. Let $l = AB$ be a line, P a point not belonging to l, and l' the parallel to l passing through P; let m denote the line AP (which cuts the parallel lines l and l' at the points A and P) and let C be a point on l' in the same half-plane w.r.t. m as B; finally, let D be a point on l' such that P is between C and D, let E be a point on l such that A is between E and B, let F be a point on m such that A is between F and P, and let G be a point on m such that P is between G and A.

Theorem 0.3.5. *In the situation described above, the following angles are congruent:*

$$\angle DPA \simeq \angle PAB \simeq \angle EAF \simeq \angle GPC,$$
$$\angle DPG \simeq \angle FAB \simeq \angle EAP \simeq \angle APC.$$

The congruence $\angle DPA \simeq \angle PAB$ follows by performing a symmetry with center at the midpoint O of $[A, P]$. The congruence

$\angle DPA \simeq \angle GPC$ follows from Theorem 0.3.4 because these angles are vertical. The other congruences are proved by the same arguments.

The next assertion (about the sum of angles of a triangle) is one of the most famous in Euclidean geometry. It is equivalent to Euclid's "Fifth Postulate" or to our Axiom V. A *triangle* $\triangle ABC$, where the three points A, B, C (called *vertices*) do not lie on the same line, is defined as the union of the three segments $[A, B]$, $[B, C]$, $[C, A]$ (called its *sides*); the angles $\angle ABC$, $\angle BCA$, $\angle CAB$ are its (*interior*) *angles.*

Corollary 0.3.6. *The sum of the angles of a triangle is two right angles.*

The proof is immediate: if we draw a parallel to BC through A and apply Theorem 0.3.4, we see (look at Figure 0.4) that the three angles fit together neatly at the vertex A, forming two right angles.

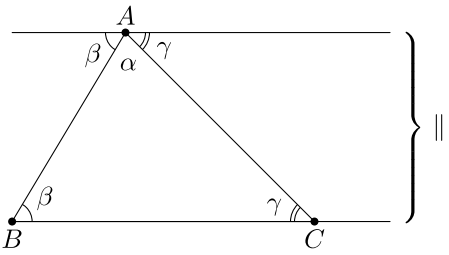


Figure 0.4. Sum of the angles of a triangle.

Note that the sum here is understood as the *geometric* sum; later, when the *measure* of angles will be introduced, we will be able to say in this situation that the *sum of measures* is equal to 180 degrees or π radians, depending on the choice of the unit of measure.

Corollary 0.3.7. *If two distinct lines are both perpendicular to the same line, then they are parallel.*

This follows from the previous corollary by arguing by contradiction.

0.3.8. Parallelograms. Given four points A, B, C, D, no three of which lie on the same line and such that any two of the segments $[A, B]$, $[B, C]$, $[C, D]$, $[D, A]$ do not intersect except at common endpoints, the union of the four segments is called a *parallelogram* provided that $AB \parallel CD$ and $BC \parallel DA$. The points A, B, C, D are said to be the *vertices* of the parallelogram, the four segments listed above are its *sides*, and the segments $[A, C]$ and $[B, D]$ are its *diagonals*. It follows from Theorem 0.3.4 that *the two pairs of opposite angles of a parallelogram $ABCD$, namely $\angle BAD$ and $\angle BCD$, $\angle ABC$ and $\angle ADC$ are congruent.* It is also true that pairs of opposite sides are also congruent, but it is more convenient to prove this later.

0.4. Parallel translations and vectors

0.4.1. Definition of parallel translations. The composition of the reflections S_l and S_m, where (l, m) is an ordered pair of parallel lines, determines an isometry called the *parallel translation* corresponding to the pair (l, m). In the case $l = m$, the corresponding parallel translation is the identity. This is not the standard way of introducing the notion of parallel translation, which as the reader surely knows, is based on the notion of vector. But we have not introduced the notion of vector yet, we do this in the next subsection.

0.4.2. Vectors. An ordered pair of points $\{O, A\}$ is said to be a *vector attached to O*; we denote it by $\overrightarrow{OA}$. Using the diagonal of a parallelogram, it is easy to define the *sum* of two vectors attached to the same point (obtaining a vector attached to the same point); using the axiom of distance, it is easy to define the *multiplication of a vector by a scalar* (i.e., by a real number), again obtaining a vector attached to the same point. It is easy to prove that *the set of all vectors attached to a fixed point in the Euclidean plane is a two-dimensional vector space.* The details are left to the reader.

Two vectors $\overrightarrow{OA}$ and $\overrightarrow{O'A'}$ are called *equal* or *equivalent* if $OAO'A'$ is a parallelogram or if one of the rays $[O, A\rangle$, $[O', A'\rangle$ contains the other and $|OA| = |O'A'|$. Equality of vectors in an equivalence relation (i.e., it is symmetric, reflexive and transitive) and therefore the set of all (attached) vectors splits into equivalence classes; each

equivalence class

$$\mathbf{v} := \left\{ \overrightarrow{OA} \,|\, \overrightarrow{OA} = \overrightarrow{O_0A_0} \right\},$$

where O_0 and A_0 are given points, is called a *free vector*. It is easy to define the *sum* of two free vectors and the *multiplication of a free vector by a scalar*, and then to prove that *the set of all free vectors in the Euclidean plane is a two-dimensional vector space.* Here again the details are left to the reader.

0.4.3. Constructing parallel translations. Let $T : \mathbb{E}^2 \to \mathbb{E}^2$ be the parallel translation corresponding to the ordered pair of parallel lines (l, l'). Let m be any perpendicular to l and let A and A' be the intersection points of m with l and l', respectively. We will call the number $|AA'|$ the *distance between the parallels l and l'*. Let us prove that this number is well defined, i.e., it does not depend on the choice of m. Let m' be another perpendicular to l, intersecting l and and l' at B and B', respectively. Then it follows from Theorem 0.3.4 and Corollary 0.3.6 that m and m' are both perpendicular to l' and parallel to each other. Now the symmetry with center O, where O is the midpoint of the diagonal AB', takes $[A, A']$ to $[B', B]$ (as can be easily proved), and therefore $|AA'| = |B'B|$, as claimed. Now if we take an arbitrary point P and denote by $P' = T(P)$ its image under the parallel translation corresponding to the ordered pair of parallel lines (l, l'), it is easy to see that $|PP'| = 2|AA'|$.

We have just shown that any (free) vector $\mathbf{v}$, in particular, the free vector determined by the attached vector $\overrightarrow{AA'}$, defines the *parallel translation* $T_{\mathbf{v}} : \mathbb{E}^2 \to \mathbb{E}^2$, the image Q of a point P being the extremity of the given vector attached to P, i.e., $T_{\mathbf{v}}(P) := Q$, where $\overrightarrow{PQ} \in \mathbf{v}$. This is the construction of parallel translations which is no doubt familiar to the reader. From this construction it immediately follows that the composition of two translations by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the translation by the vector $\mathbf{v}_1 + \mathbf{v}_2$.

Theorem 0.4.4. *Any parallel translation is an isometry. The composition of two translations is a translation, namely the translation by the sum of the two vectors defining the given translations.*

The first statement of the theorem follows from 0.3.1(vi), the second one was just established above.

0.5. Triangles: congruence, properties

0.5.1. Congruence tests for triangles. In this subsection, using the properties of parallel translations, rotations and reflections, we obtain the three classical tests of the congruence of triangles. These tests have different names: in continental Europe, they are simply numbered ("first test", "second test", etc.), in the US the more informative notation SAS, ASA, AAA is used (for example, SAS stands for side-angle-side). In all three cases, we consider two given triangles: $\triangle ABC$ and $\triangle A'B'C'$, with the following notation for the angles $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$ of triangle $\triangle ABC$, and similar notation α', β', γ' for those of $\triangle A'B'C'$.

Theorem 0.5.2 (SAS)**.** *If a given triangle has one angle and the two sides limiting it congruent to an angle and to the sides limiting that angle in a second triangle, then the two triangles are congruent.*

In our notation, the assumptions of the theorem may be written as

$$\alpha \simeq \alpha', \quad [A,B] \simeq [A',B'], \quad [A,C] \simeq [A',C'].$$

The parallel translation by the vector $\overrightarrow{AA'}$ followed by an appropriate rotation will send $[A,B]$ onto $[A',B']$; let C'' denote the image of C under the composition of the above translation and rotation. Then two cases can arise, depending on which half-plane with respect to $A'B'$ contains C''. If C'' is in the same half-plane as C', then these two points coincide and we are done. If not, we first perform a reflection in the line $A'B'$, after which the image of C'' will coincide with C'. Note that here we are consistently using the fact that the transformations considered are isometries.

Theorem 0.5.3 (ASA)**.** *If a given triangle has one side and the two angles with vertices at its endpoints congruent to a side and to the angles with vertices at the endpoints of that side in a second triangle, then the two triangles are congruent.*

The proof of this theorem, as well as that of the next one, is similar to the proof of Theorem 0.5.2.

Theorem 0.5.4 (SSS)**.** *If a given triangle has three sides respectively congruent to three sides of a second triangle, then the two triangles are congruent.*

Remark 0.5.5. There is no "fourth congruence test" (ASS): it is *not* always true that if two triangles have one congruent angle and two congruent sides, then they are congruent, as Figure 0.5 shows.

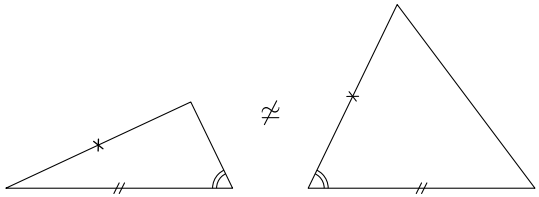


Figure 0.5. Noncongruent triangles.

Remark 0.5.6. There is another version of the congruence tests: instead of assuming the *congruence* of sides and/or angles, one assumes the *equality* of the lengths of the sides and the equality of the measures of the angles. We cannot do this at this stage, because the *measure of angles* has not been defined yet.

0.5.7. Special lines in triangles and special triangles. Given an arbitrary triangle $\triangle ABC$, by drawing the perpendiculars from the vertices to the opposite sides, we obtain three segments $[A, H]$, $[B, I]$, $[C, J]$, which are called the *altitudes* of the triangle. The lines AR, BS, CT that divide the angles of $\triangle ABC$ into pairs of congruent angles are called the *bisectors* of $\triangle ABC$. The lines AM, BN, and CP joining the vertices to the midpoints M, N, and P of the opposite sides are its *medians* (see Figure 0.6).

In any triangle, the three altitudes intersect in one point, and so do the medians and the bisectors; this will be proved later.

A triangle is called *isosceles* if two of its sides are congruent (these two sides are called *lateral*, the third one is the *base*), and *equilateral* if all three sides are congruent.

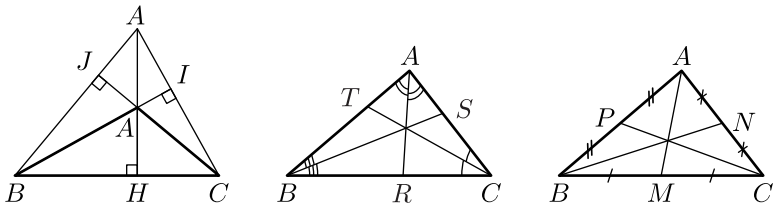


Figure 0.6. Altitudes, bisectors, and medians of a triangle.

Theorem 0.5.8. *In an isosceles triangle, the angles at the base are congruent, and the median to the base coincides with the corresponding altitude and bisector.*

To prove this, one constructs the median $[A, M]$ to the base and applies the third congruence test to $\triangle ABM$ and $\triangle ACM$.

Theorem 0.5.9. *In an equilateral triangle all three angles are congruent, and all the medians coincide with the corresponding altitudes and bisectors.*

This follows from the previous theorem.

0.6. Homothety and similitude

0.6.1. Homothety. A *homothety* with *center* O and *ratio* $\rho > 0$ is the transformation that assigns to each point P the extremity P' of the vector given by

$$\overrightarrow{OP'} := \rho \cdot \overrightarrow{OP}.$$

Figures obtained from each other by homothety are called *homothetic*. A homothety of ratio $\rho = 1$ is the identity transformation; if $\rho > 1$, then the size of the image of a figure increases but its shape remains the same, and if $\rho < 1$, then the shape still does not change, but the size decreases. The definition of homothety (and the axioms) immediately imply the following properties of homothety.

Proposition 0.6.2. *Any homothety*

(i) *takes lines to lines, segments to segments, rays to rays;*

(ii) *takes parallel lines to parallel lines;*

(iii) *takes perpendiculars to perpendiculars, angles to congruent angles;*

(iv) *takes triangles to triangles;*

(v) *takes circles to circles.*

0.6.3. Similitude. Any composition of isometries and homotheties is called a *similitude.* Two figures obtained from each other by a similitude are called *similar.* Similarity of figures corresponds to the intuitive notion of having the same shape. The above definition implies that *any similitude possesses properties* (i)–(v) *listed in Subsection 0.3.1*, that *the composition of similitudes is a similitude*, and that any similitude possesses a *ratio*, i.e., a coefficient $\rho > 0$ such that $|P'Q'| = \rho \cdot |PQ|$, where P, Q are arbitrary points and $P'Q'$ are their images under the given similitude.

The following theorems can be called similarity tests for triangles.

Theorem 0.6.4. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|} = \text{const} =: \rho$$

are similar.

To prove this, one subjects triangle $\triangle A'B'C'$ to a homothety of ratio ρ and arbitrary center, and then applies the third congruence test for triangles (Theorem 0.5.4, SSS).

Theorem 0.6.5. *If two triangles have all three angles congruent, then they are similar.*

Actually, it suffices to require that *two* angles be congruent, because then the third angles are congruent automatically (by Corollary 0.3.6). If the angles at the vertices A, B, C of $\triangle ABC$ are, respectively, congruent to the angles at the vertices A', B', C' of $\triangle A'B'C'$, then a homothety with the ratio $\rho := |AB|/|A'B'|$ followed by an application of the second congruence test proves the theorem.

0.6.6. Right triangles. A triangle is said to be a *right triangle* if one of its angles is a right angle; the side opposite to the right angle is known as the *hypotenuse*, the other two sides are the *legs*. Since

the sum of angles of a triangle is two right angles, it follows that the other two angles are *acute*, i.e., their measure (see 0.7.2) is less than $\pi/2$. The following statement is an immediate consequence of Theorem 0.6.5.

Corollary 0.6.7. *If a right triangle has an acute angle congruent to an acute angle of another right triangle, then the two triangles are similar.*

We conclude this subsection with one of the most famous theorems in geometry, attributed to Pythagoras, although it was known to the Egyptians and Babylonians long before Pythagoras' time.

Theorem 0.6.8 (Pythagoras, 6th century B.C.)**.** *In a right triangle, the square of* (*length of*) *the hypotenuse is equal to the sum of the squares of* (*the lengths of*) *the legs.*

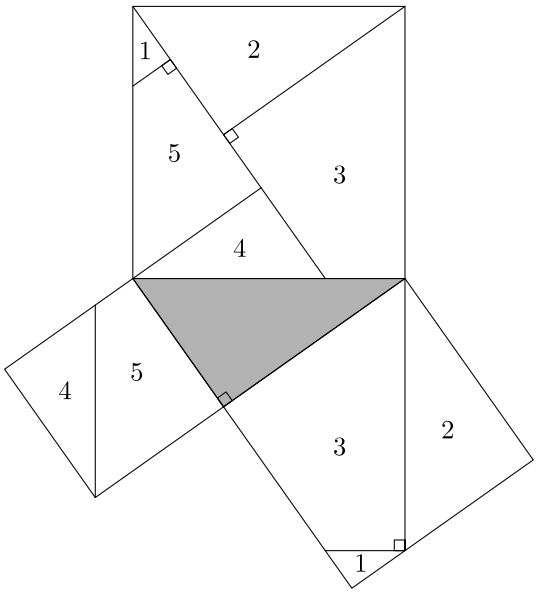


Figure 0.7. Pythagoras' pants.

A popular proof of this theorem is shown in Figure 0.7. But this proof (known to Russian students as "Pythagoras' pants") is based

on the notion of area, which we have not introduced yet. There are many other proofs of this theorem; the reader will find 93 (!) proofs in a web site easily accessed by googling.

Here we describe a simple proof based on similitude. Let $\triangle ABC$ be our triangle with right angle at C, angles α at A and β at B, hypotenuse c and legs a and b. Drop the perpendicular CH from C to AB, where $H \in [A, B]$; denote $h := |CH|$, denote $c_1 := |AH|$ and $c_2 := |HB|$. Then we obtain two right triangles similar to $\triangle ABC$, so that we have $b/c_1 = c/b$ and $a/c_2 = c/a$, whence $a^2 = cc_2$ and $b^2 = cc_1$. Adding these two equalities and using the fact that $c = c_1 + c_2$, we obtain

$$a^2 + b^2 = c^2.$$

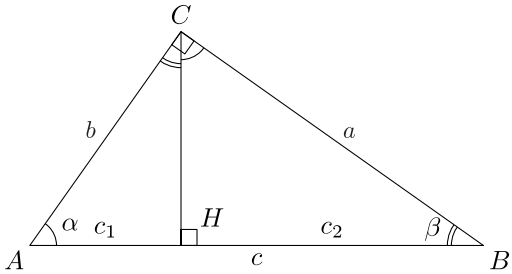


Figure 0.8. Proof of the Pythagoras theorem.

0.7. Angle measure and trigonometry

0.7.1. Measuring angles in degrees. The traditional angle measure comes from navigation, the unit of measure being the degree. If we divide a circle into 360 congruent arcs and consider the angle determined by two rays issuing from the center O of the circle and passing through the endpoints of one of these arcs, we obtain a one *degree* (1°) angle; if we take, say, the endpoints of seventeen such successive arcs $A_1, A_2, \ldots, A_{18}$, then the measure of the angle $\angle A_1 O A_{18}$ will be 17°, and so on. Right angles measure 90°, so that 180° angles, for which the rays lie on one line, are those that in the times of Euclid were referred to as forming "two right angles". Degrees are

subdivided into "minutes" and minutes into "seconds", but we will not need these finer measures.

0.7.2. Measuring angles in radians. A more modern and more convenient unit of measure from the mathematical point of view is the radian. Usually, it is defined in calculus courses, but here we define the *radian* as the unit of angle measure proportional to the degree and such that a 180° angle is the same as an angle of π radians. Mathematicians prefer radians to degrees and usually say, for example, "this angle equals π" rather than "this angle measures 180°".

0.7.3. Trigonometry of the triangle. In elementary geometry (and in our exposition here) only angles of nonnegative measure less than or equal to 180° are considered, and accordingly the trigonometric functions will be defined only for angles in triangles (the more general case of trigonometric functions for arbitrary values of the argument is usually studied in calculus courses). Given a right triangle $\triangle HAB$ with hypotenuse $[A, B]$ of length h and legs $[H, A]$ and $[H, B]$ of length a and b, we define the trigonometric functions *sine*, *cosine*, and *tangent* of the angle $\alpha = \angle BAH$ as the following ratios:

$$\sin\alpha := \frac{b}{h}, \quad \cos\alpha := \frac{a}{h}, \quad \tan\alpha := \frac{b}{a}.$$

These definitions immediately imply that

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \quad \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha, \quad \sin^2\alpha + \cos^2\alpha = 1.$$

In an arbitrary triangle ABC (with angles α, β, γ and opposite sides of lengths a, b, c), there are classical relationships between the lengths of the sides and the sines of the angles (or their cosines).

Theorem 0.7.4 (Sine Theorem)**.**

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

To prove this, draw the perpendicular CH to AB, where H is on the line AB; let $h := |CH|$; we obtain two right triangles which (by the definition of sine) give $\sin\alpha = h/b$ and $\sin\beta = h/a$ (see Figure 0.9(a), which shows the case in which $H \in [A, B]$; in the

case shown in (b), the same equalities are obtained in a slightly different way). Simple manipulations with these two equalities yield $(\sin\alpha)/a = (\sin\beta)/b$. The second equality is proved similarly.

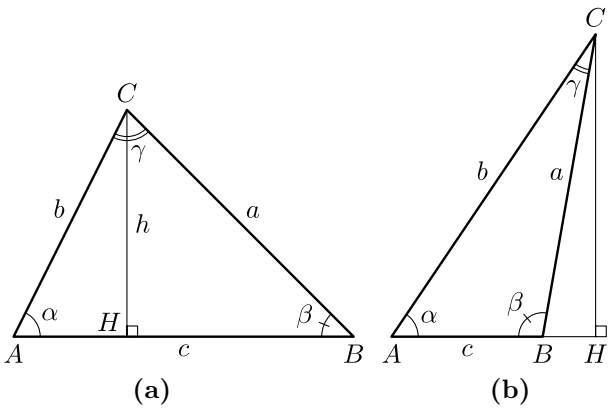


Figure 0.9. Proving the sine and cosine theorems.

Theorem 0.7.5 (Cosine Theorem)**.**

$$a^2 = b^2 + c^2 - 2\,b\,c\cos\alpha.$$

Using the same construction as in the previous proof, denote $c_1 := [A, H]$ and $c_2 := [H, B]$ (so that $c = c_1 + c_2$), and apply the Pythagoras theorem to the two right triangles (Figure 0.9(a)), obtaining $a^2 = c_2^2 + h^2$ and $b^2 = c_1^2 + h^2$; the definition of the cosine yields $\cos\alpha = c_1/b$. Now simple manipulations with these three equalities give the required result. In the case shown in Figure 0.9(b), the result is obtained in a slightly different way.

0.8. Properties of the circle

Recall that the *circle* of *center* O and *radius* $r > 0$ is the set of all points M such that $|OM| = r$. If A and B are points on the circle, then the segment $[A, B]$ is said to be a *chord*. If $[A, B]$ contains the center O, then O is obviously the midpoint of $[A, B]$ and $[A, B]$ is called a *diameter* of the circle. If $[A, B]$ is a chord and C is a point on the circle, we say that the angle ACB *subtends* the chord $[A, B]$.

Theorem 0.8.1. *The line joining the midpoint of a chord to the center of the circle is perpendicular to the chord.*

Let M be the midpoint of the chord $[A, B]$ (Figure 0.10(a)); then the two triangles OAM and OBM are congruent by the third congruence test SSS (Theorem 0.5.4), hence the two angles at M are congruent and therefore both must be right.

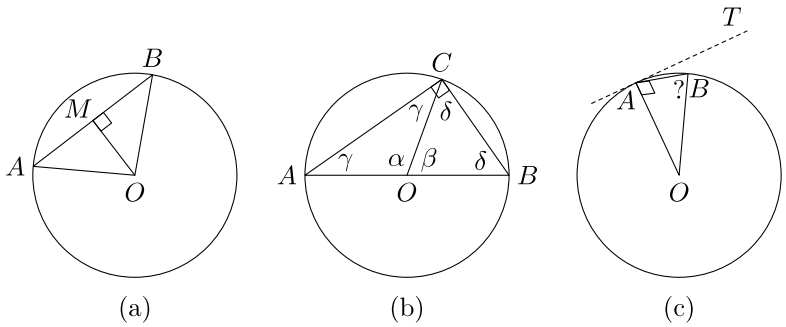


Figure 0.10. Right angles in the circle.

Theorem 0.8.2. *Any angle subtending a diameter of a circle is a right angle.*

Let $[A, B]$ be a diameter and C a point on the circle (Figure 0.10(b)). Join C and O and denote by α and β the two angles formed at O. Then $\alpha + \beta = \pi$. Let γ be the measure of the two congruent angles of the isosceles triangle OAC, and δ the same measure in triangle OBC. Then $\alpha + 2\gamma = \pi$ and $\beta + 2\delta = \pi$ (by Corollary 0.3.6). Extracting α and β from these two equalities and substituting into $\alpha + \beta = \pi$, we obtain $\gamma + \delta = \pi/2$, as required.

A line is called *tangent* to a circle if it has only one common point with the circle.

Theorem 0.8.3. *Any line passing through a point on a circle and perpendicular to the radius passing through that point is tangent to the circle.*

Suppose it isn't, and let B be the intersection point of the line with the circle other than A, the extremity of the given radius (Figure 0.10(c)). Then triangle OAB is isosceles and so has right angles both at A and at B, which means that its angle sum is greater than two right angles, in contradiction with Corollary 0.3.6 (see Figure 0.12(b)).

Theorem 0.8.4. *Suppose two lines pass through a point T outside a circle centered at O and are tangent to it at the points A and B. Then the segments $[T, A]$ and $[TB]$ are congruent, and the ray $[T, O\rangle]$ is the bisector of angle $\angle ATB$, i.e., the angles $\angle ATO$ and $\angle BTO$ are congruent.*

This readily follows from the congruence of triangles TAO and TBO (see Figure 0.11).

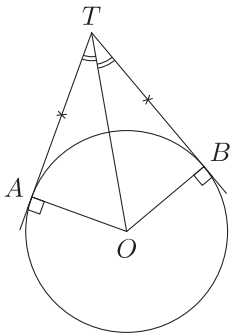


Figure 0.11. Tangents to a circle.

A circle is said to be *inscribed* in a triangle if it is tangent to all three of its sides, and *circumscribed* to it if it passes through the triangle's vertices.

Theorem 0.8.5. *All three bisectors of any triangle intersect in one common point, and this point is the center of the circle inscribed in the triangle.*

This immediately follows from Theorem 0.8.4 (see Figure 0.12(a)).

Theorem 0.8.6. *All three perpendiculars to the sides of a triangle that pass through the midpoints of the sides intersect in one common point, and this point is the center of the circle circumscribed to the triangle.*

This immediately follows from Theorem 0.8.1.

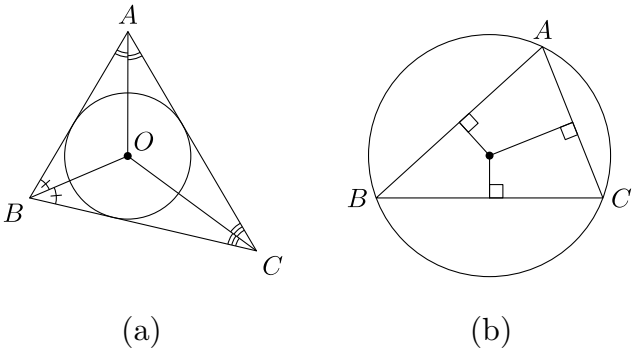


Figure 0.12. Inscribed and circumscribed circles.

Theorem 0.8.7. *Congruent chords in a circle are subtended by congruent angles.*

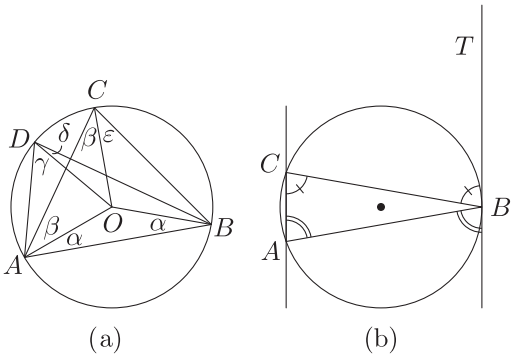


Figure 0.13. Congruent angles in the circle.

Suppose D and C are points of the circle, AB is its chord, and assume that D and C lie on the same side of line AB (this last assumption is implicitly included in the statement of the theorem). Join the four points A, B, C, D to the center O of the circle, thereby obtaining five isosceles triangles, and denote their equal angles by $\alpha, \beta, \gamma, \delta, \varepsilon$, as shown in Figure 0.13(a). Since the angle sum of triangles ABC and ABD is π, we have

$$2\alpha + 2\beta + 2\varepsilon = \pi, \quad 2\alpha + 2\gamma + 2\delta = \pi.$$

Subtracting the second equality from the first and dividing by 2, we obtain $\beta + \varepsilon = \gamma + \delta$, as required.

Theorem 0.8.8. *Let the angle $\angle ACB$ subtend the chord $[A, B]$ of a circle, let BT be the tangent to the circle at B, and assume that T is on the same side of the line AB as C (see Figure* 0.13(b)*). Then the angles $\angle ACB$ and $\angle CBT$ are congruent.*

By the previous theorem, we can assume, without loss of generality, that AC is parallel to BT. Then the required congruence follows from one of the properties of secants of two parallel lines (Theorem 0.3.4).

0.9. Isometries of the plane

Recall that an isometry of the plane is a transformation of the plane that preserves distances. Any isometry is a bijection of the plane onto itself. Examples of isometries that we have encountered are parallel translations, rotations, and reflections in a line. Another type of isometry is described in the next subsection.

0.9.1. Glide symmetries. By definition, a *glide symmetry* $GF(l, \overrightarrow{v})$ is the composition of a reflection in the line l and the parallel translation by the vector $\overrightarrow{v}$, where $\overrightarrow{v}$ is parallel to l. As can be seen from the figure, glide symmetries reverse orientation. If $\overrightarrow{v} = 0$, then of course the glide symmetry $GF(\overrightarrow{v})$ is simply the reflection in the line l.

The next theorem, the main theorem on the structure of isometries of the plane, says that there are no isometries other than the three mentioned above.

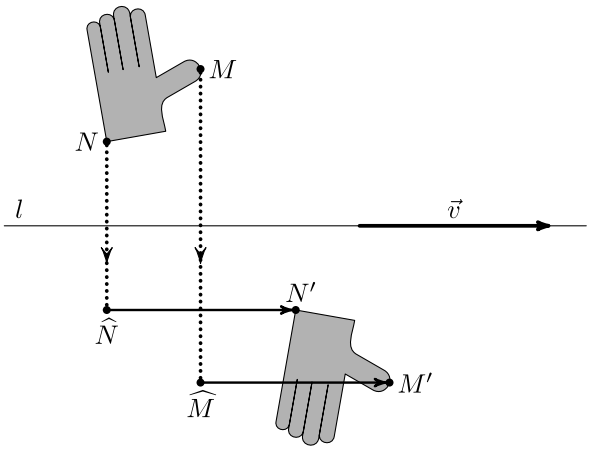


Figure 0.14. Glide symmetry.

Theorem 0.9.2. *Any isometry of the plane is a parallel translation, or a rotation, or a glide symmetry.*

Let OAB be an orthonormal frame (this means that OA is perpendicular to OB and $|OA| = |OB| = 1$); then the position of any point M in the plane is determined by its Cartesian coordinates (x, y) w.r.t. this frame. Denote by $O'A'B'$ the image of OAB and by M' the image of M under the given isometry. Then the point M' has the coordinates (x, y) w.r.t. $O'A'B'$. This means that *any isometry is entirely determined by an orthonormal frame and its image.*

So let $O'A'B'$ be the image of the orthonormal frame OAB. We will consider several cases.

First we consider the case in which the lines OA and $O'A'$ are not parallel. Then they intersect at a point P. Construct the circle circumscribed to $\triangle OO'P$ and denote by Q the intersection point of this circle with the perpendicular to OO' from its midpoint, where Q lies on the same side of OO' as P. Then the rotation by the angle $([QO\rangle, [QO'\rangle)$ takes O to O' and A to A' (because angles $\angle QOP$ and $\angle QO'P$ are congruent as subtending the same chord $[Q, P]$, see Theorem 0.8.7).

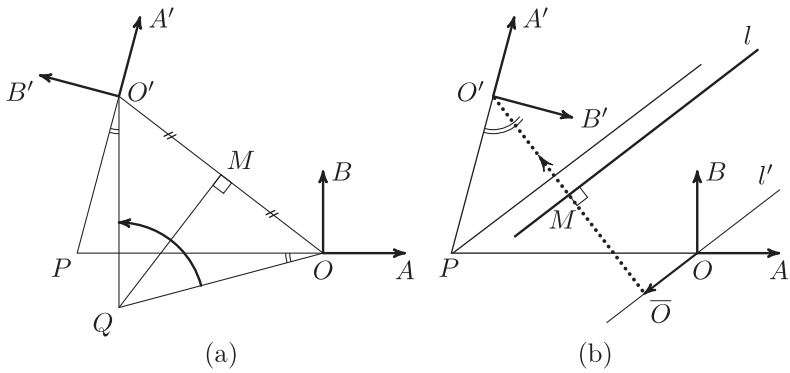


Figure 0.15. $\overrightarrow{OA} \nparallel \overrightarrow{O'A'}$: rotation and glide symmetry.

Now there are two subcases possible (see Figure 0.15(a) and (b)): either this rotation takes B to B' (in this case the theorem is proved, because the rotation takes the given frame to its image by the given isometry, and so that isometry is a rotation, namely the one just constructed) or it takes B to $\hat{B}$, where $\hat{B}$ is the point symmetric to B' w.r.t. the line $O'A'$.

In the second subcase we perform a different construction. We draw the bisector of $\angle OPO'$ through the point P and construct the parallel l' to this bisector through the point O'. Let $\bar{O}$ be the foot of the perpendicular drawn from O to l', let M be the midpoint of $[O, \bar{O}]$; denote by l the parallel to l' through M. Then it is easy to see that the glide symmetry $\left(l, \overrightarrow{v}\right)$, where $\overrightarrow{v} = \overrightarrow{\bar{O}O'}$, takes the frame OAB to $O'A'B'$, which proves the theorem in this subcase as well.

Now we pass to the case in which the lines OA and $O'A'$ are parallel.

Denote

$$e_1 := \overrightarrow{OA}, \quad e_2 := \overrightarrow{OB}, \quad e_1' := \overrightarrow{O'A'}, \quad e_2' := \overrightarrow{O'B'}.$$

Let us consider the following four subcases:

(1) the vectors of both pairs (e_1, e_1') and (e_2, e_2') point in opposite directions;

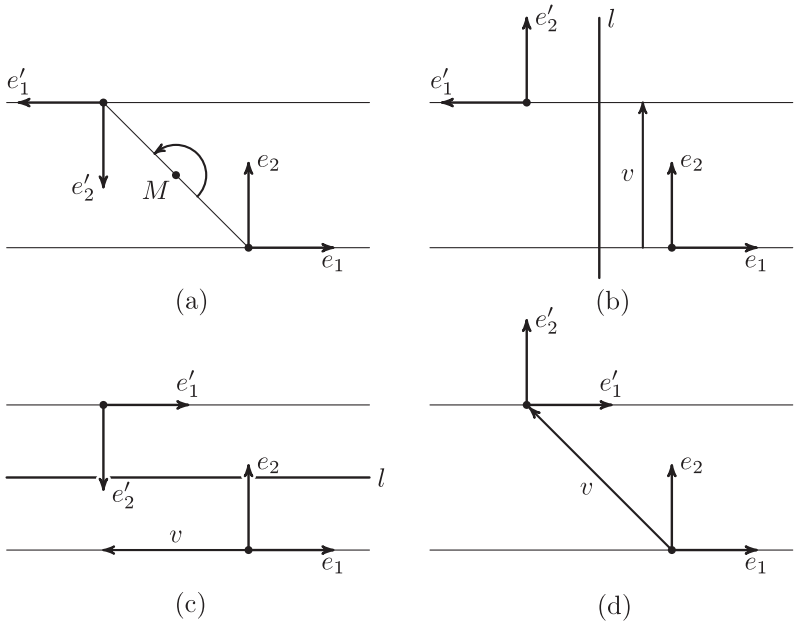


Figure 0.16. $\overrightarrow{OA} \parallel \overrightarrow{O'A'}$: rotation, glide symmetries or parallel translation.

(2) the vectors of the first pair point in opposite directions, and those of the second pair, in the same direction;

(3) the vectors of the first pair point in the same direction, and those of the second pair, in opposite ones;

(4) the vectors of both pairs point in the same directions.

In subcase (1), the rotation by π about the midpoint of $[O, O']$ will take the frame OAB to $O'A'B'$ (Figure 0.16(a)), in subcases (2) and (3) this is achieved by glide symmetries (Figure 0.16(b), (c)), and finally in subcase (4), by the parallel translation by the vector $\overrightarrow{OO'}$ (Figure 0.16(d)).

0.10. Space geometry

In this subsection, we do not intend to develop Euclidean space geometry (or even to summarize its main constructions and facts). We only indicate a (very natural, but superfluous) system of axioms for Euclidean space geometry and mention some facts related to isometries in 3-space, omitting all proofs and not going into detail.

0.10.1. Axioms for Euclidean space geometry. There are four types of undefined notions in our theory: *points*, (*straight*) *lines*, *planes*, and (*Euclidean*) *space*. Points are denoted by various capital letters ($A, B, C, \dots, P, Q, \dots$, sometimes supplied with subscripts or superscripts), lines are denoted by small italics ($l, m, n, \dots$, also possibly endowed with subscripts or superscripts), planes, by capital script letters ($\mathcal{P}, \mathcal{Q}, \mathcal{R}, \dots$), while space is denoted by $\mathbb{E}^3$. These objects satisfy the following ten axioms.

I$_S$. *The space* $\mathbb{E}^3$ *is the set of all points.*

II$_S$. *All lines and all planes are nonempty subsets of the space* $\mathbb{E}^3$.

III$_S$. *The distance* $d(A, B)$ *between any two points of space is defined, and it satisfies the same four conditions* (i)–(iv) *as in plane geometry.*

IV$_S$. *Each plane is a Euclidean plane, i.e, points and lines in it satisfy all the axioms of Euclidean plane geometry; the distance function in all the planes is the same as the one in axiom* III$_S$ *above.*

Note that the definition of parallel line used in axiom V of plane geometry must be modified: two lines are called *parallel in space* if they coincide or lie in the same plane and have no common points. Two planes in space are called *parallel* if they coincide or have no common points.

V$_S$. *Given a plane* $\mathcal{P}$ *and a point* A*, there exists a unique plane containing* A *and parallel to* $\mathcal{P}$.

In a sense this axiom says that space is of dimension no greater than 3; the next one says its dimension is at least 3.

VI$_S$. *For any plane there exists at least one point* $P \in \mathbb{E}^3$ *not contained in that plane.*

VII$_S$. *There exists a plane passing through any three points, and if these points are not contained in a line, this plane is unique.*

VIII$_S$. *If two distinct points lie in a plane, then the line passing through them is contained in the plane.*

Using these axioms, it is easy to prove that any two intersecting lines determine a unique plane containing them. A line l is called *perpendicular* to a plane $\mathcal{P}$ if it has a common point H with $\mathcal{P}$ and is perpendicular to any line passing through H and contained in $\mathcal{P}$.

IX$_S$. *For any plane* $\mathcal{P}$ *and any point* $A \in \mathbb{E}^3$, *there exists a unique perpendicular to* l *containing* A.

Two points A and B are said to *lie on the same side* of a given plane $\mathcal{P}$ if the segment $[A, B]$ does not intersect $\mathcal{P}$, and *lie in opposite sides* if it does. Thus any plane $\mathcal{P}$ determines two *open half-spaces*, each consisting of points lying on the same side of the plane, while any two points from different half-spaces lie on opposite sides of $\mathcal{P}$. (A *closed half-space* is defined as the union of an open half-space with the plane that bounds it.) A *reflection* in a plane $\mathcal{P}$ is the transformation that takes any point A not in $\mathcal{P}$ to the point A' that lies on the line AH perpendicular to $\mathcal{P}$, with $H \in \mathcal{P}$, so that H is the midpoint of AA', and takes any point $P \in \mathcal{P}$ to itself.

X$_S$. *The reflection in any plane is an isometry.*

It follows immediately from the definition of reflection (and the axioms) that *any reflection in* $\mathcal{P}$ *interchanges the two half-planes determined by* $\mathcal{P}$.

Remark. Actually, axioms IX$_S$ and X$_S$ can be derived from the other axioms, in particular, from the very strong axiom IV$_S$. We have included them for the sake of brevity and in order to stress the analogy between two-dimensional and three-dimensional Euclidean geometry.

0.10.2. Convex polyhedra. A subset $\mathcal{C}$ of $\mathbb{E}^3$ (or of the plane $\mathbb{E}^2$) is called *convex* if $A, B \in \mathcal{C}$ implies that $[A, B] \subset \mathcal{C}$. It immediately follows from definitions that any open half-space is convex, and so is any closed half-space (the same is true for half-planes).

Proposition 0.10.3. *The intersection of any two convex sets is convex.*

The intersection of a finite number of closed half-spaces is said to be a *convex polyhedron* provided that it is a bounded set and contains interior points (i.e., points that are the centers of balls entirely lying in it). The reader is undoubtedly familiar with such convex polyhedra as the *cube*, the *parallelepiped*, various *tetrahedra*, certain *prisms*, and has possibly heard of regular polyhedra, e.g., of the *octahedron* or the *dodecahedron*.

0.10.4. Isometries in 3-space. Recall that an *isometry* of $\mathbb{E}^3$ is a transformation of $\mathbb{E}^3$ that preserves distances. Any isometry is a bijection of $\mathbb{E}^3$. It can be proved that *any isometry is the composition of reflections*. (In this subsection by reflection we always mean reflection in a plane.) Let us list some important examples of isometries.

(i) *Parallel translations.* The composition τ of two reflections in parallel planes $\mathcal{P} \parallel \mathcal{Q}$ is said to be a *parallel translation*. Just as in the planar case, each parallel translation is determined by a *free vector*, namely the vector $2\overrightarrow{HK}$, where the line HK is perpendicular to the reflection planes and $H \in \mathcal{P}$, $K \in \mathcal{Q}$.

(ii) *Rotations about an axis.* The composition ρ of two reflections in intersecting planes $\mathcal{P} \nparallel \mathcal{Q}$ is said to be a *rotation about an axis*, namely about the line $l := \mathcal{P} \cap \mathcal{Q}$. It is easy to show that the restriction of ρ to any plane $\mathcal{R}$ perpendicular to l is a rotation of that plane about the point $O := \mathcal{R} \cap l$ by an angle equal to twice the angle between $\mathcal{P}$ and $\mathcal{Q}$. (The definition of the (dihedral) angle between two planes is left to the reader, as well as the determination of the direction of rotation.)

A particular case of rotation about an axis is *axial symmetry*, which is the composition of two reflections in perpendicular planes, and may be described as follows: the image A' of any point A is

obtained by constructing the perpendicular AH from A to the axis of rotation l, $l \ni H$, and extending $[A, H]$ to $[A, A']$, where H is the midpoint of $[A, A']$. It can also be described as a rotation about l by $180°$.

(iii) *Central symmetries.* A *central symmetry* σ with center C is the transformation that takes any point A to the point A' symmetric to A with respect to C, i.e., to the point A' such that C is the midpoint of $[A, A']$. Another way of looking at σ is to define it as the composition of three reflections in three pairwise perpendicular planes with common point O.

(iv) *Helicoidal motions.* A *helicoidal motion* χ is the composition of a rotation about an axis and a parallel translation in the direction of the axis. Thus χ is the composition of four reflections. A helicoidal motion models the motion of a screw being screwed into a piece of wood, and therefore is often encountered in mechanics.

Theorem 0.10.5. *Any isometry of* $\mathbb{E}^3$ *is a composition of reflections.*

The proof is not very difficult and is similar to that of Theorem 0.9.2, i.e., it consists of a case by case study of the image of an orthonormal frame under the given isometry.

0.10.6. Orientation. Let $(e_1, e_2, e_3) = \left(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\right)$ be an orthonormal frame, i.e., an ordered triple of pairwise perpendicular unit vectors with common origin O. Another such frame (e_1', e_2', e_3') has the *same orientation* as (e_1, e_2, e_3) if it can be taken to the other by the composition of an even number of reflections, and the *opposite orientation* if it can be taken to the other by an odd number of reflections. Thus (e_1, e_2, e_3) and $(e_1, e_2, -e_3)$ have opposite orientations. It is easy to see that the set of all orthonormal frames splits into two equivalence classes with respect to the relation "to have the same orientation". The choice of one of these classes is called a choice of *orientation.*

In physics courses (and sometimes in mathematics courses), the notion of "positively oriented" frame is introduced, along with such expressions as the "right-hand rule" (e.g. in electrodynamics). Actually, there is no *mathematical* way to choose canonically a "preferred"

orientation among the two existing ones, so that the expression "positively oriented frame" is mathematically meaningless (just as the expressions "rotation in the positive direction" or "clockwise rotation" in plane geometry.)

Isometries of $\mathbb{E}^3$ that preserve orientation are called *motions*. Any motion is the composition of an even number of reflections. Isometries that do not preserve orientation are said to be *orientation-reversing*.

0.10.7. Compositions of isometries. The following properties of the composition of specific types of isometries are not hard to prove:

(i) *the composition of two parallel translations is a parallel translation;*

(ii) *the composition of two rotations about intersecting axes is a rotation about an axis passing through the common point of the two given axes;*

(iii) *the composition of two reflections is a parallel translation or a rotation depending on whether the reflection planes are parallel or not;*

(iv) *the composition of two central symmetries is a parallel translation;*

(v) *the composition of two motions is a motion;*

(vi) *the composition of two orientation-reversing isometries is a motion.*

In Chapters 1 and 3 of this book, we will be interested in isometries related to various specific two- and three-dimensional objects (such as the square, the cube, the regular tetrahedron), and we will need to be able to find, very concretely, what the composition of two given specific isometries actually is (for instance, find an effective construction of the axis of rotation in the case of item (ii) above).

Chapter 1

Toy Geometries and Main Definitions

In this chapter, we study five toy examples of geometries (symmetries of the equilateral triangle, the square, the cube, the circle and the sphere) and a model of the geometry of the so-called elliptic plane. These examples prepare us for the main definition (given in Sec. 1.4) of this course: a geometry in the sense of Klein is a set with a transformation group acting on it. Before that, we present some useful general notions related to transformation groups. Further, we study the relationships (called morphisms or equivariant maps) between different geometries, thus introducing the category of all geometries. The notions introduced in this chapter are illustrated by some problems (dealing with toy models of geometries) collected at the end of the chapter.

But before we begin with these topics, we briefly recall some terminology from elementary Euclidean geometry.

1.1. Isometries of the Euclidean plane and space

We assume that the reader is familiar with the basic notions and facts of Euclidean geometry in the plane and in space (these notions and facts are summarized in Chapter 0). One can think of Euclidean geometry as an axiomatic theory (not too rigorously taught in high

school) or as a small chapter of linear algebra (the plane $\mathbb{R}^2$ and the space $\mathbb{R}^3$ supplied with the standard metric). It is irrelevant to us which of these two points of view is adopted by the reader, and the aim of this subsection is merely to fix some terminology.

An *isometry* of the Euclidean plane $\mathbb{R}^2$ (or space $\mathbb{R}^3$) is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (respectively, $f : \mathbb{R}^3 \to \mathbb{R}^3$) which preserves the distance d between points, i.e.,

$$d(f(P), f(Q)) = d(P, Q)$$

for any pair of points P, Q of the plane (respectively, of space). There are two types of isometries: those which preserve orientation (they are called *motions*, or sometimes *rigid motions*) and those that reverse orientation (*orientation-reversing isometries*).

In the plane, examples of motions are *parallel translations* (determined by a fixed *translation vector*) and *rotations* (determined by a pair (C, α), where C is the *center of rotation* and α is the *oriented angle of rotation*. In space, examples of motions are *parallel translations* and *rotations* (about an axis). Rotations in space are determined by pairs (l, α), where l is the *axis of rotation*, i.e., a straight line with a specified direction on it, and α is the *angle of rotation*. The rotation (l, α) maps any point M in space to the point M' obtained by rotating M in the plane Π perpendicular to l and passing through M by the angle α in a chosen direction. The direction of rotation is of course the same in each plane parallel to Π; it must be either clockwise or counterclockwise if one looks at the plane from "above", i.e., from some point of l obtained from the point $l \cap \Pi$ by moving in the direction of the axis.

Examples of orientation-reversing isometries in the plane are *reflections* (i.e., symmetries with respect to a line). In space, examples of orientation-reversing isometries are given by *mirror symmetries* (i.e., reflections with respect to planes) and *central symmetries* (i.e., reflections with respect to a point).

All other isometries of the Euclidean plane and space are *compositions* of those listed above.

The reader who is uncomfortable with the notions appearing in this subsection is invited to look at the relevant parts of Chapter 0.

1.2. Symmetries of some figures

1.2.1. Symmetries of the equilateral triangle. Consider all the isometries of the equilateral triangle $\triangle = ABC$, i.e., all the distance-preserving mappings of this triangle onto itself. (To be definite, we assume that the letters A, B, C have been assigned to vertices in counterclockwise order.) Denote by s_A, s_B, and s_C the reflections in the bisectors of angles A, B, C of the triangle. Denote by r_0, r_1, r_2 the counterclockwise rotations about its center of gravity by 0, 120, 240 degrees, respectively. Thus r_1 takes the vertex A to B, B to C, and C to A. These six transformations are all called *symmetries* of triangle ABC and the set that they constitute is denoted by $\mathrm{Sym}(\triangle)$. Thus

$$\mathrm{Sym}(\triangle) = \{r_0,\, r_1,\, r_2,\, s_A,\, s_B,\, s_C\}.$$

There are no other isometries of $\triangle$. Indeed, any isometry takes vertices to vertices, and each one-to-one correspondence between vertices entirely determines the isometry. (For example, the correspondence $A \to B$, $B \to A$, $C \to C$ determines the reflection s_C.) But there are only six different ways to assign the letters A, B, C to three points, so there cannot be more than 6 isometries of $\triangle$.

In a certain sense, $\mathrm{Sym}(\triangle)$ is the same thing as the family of all permutations of the three letters A, B, C; this remark will be made precise in the next chapter.

We will use the symbol $*$ to denote the *composition* (or *product*) of isometries, in particular, of elements of $\mathrm{Sym}(\triangle)$, and understand expressions such as $r_1 * s_A$ to mean that r_1 is performed first, and then followed by s_A. Obviously, when we compose two elements of $\mathrm{Sym}(\triangle)$, we always obtain an element of $\mathrm{Sym}(\triangle)$.

What element is the composition of two given ones can be easily seen by drawing a picture of the triangle ABC and observing what happens to it when the given isometries are successively performed, but this can also be done without any pictures: it suffices to follow the "trajectory" of the vertices A, B, C. Thus, in the example $r_1 * s_A$, the rotation r_1 takes the vertex A to B, and then B is taken to C by the symmetry s_A; similarly, $B \to C \to B$ and $C \to A \to A$, so that the vertices A, B, C are taken to C, B, A in that order, which means that $r_1 * s_A = s_B$.

The order in which symmetries are composed is important, because the resulting symmetry may change if we inverse the order. Thus, in our example, $s_A * r_1 = s_C \neq s_B$ (as the reader will readily check), so that $r_1 * s_A \neq s_A * r_1$. So for elements of Sym($\triangle$), composition is *noncommutative*.

The compositions of all possible pairs of symmetries of $\triangle$ can be conveniently shown in the following *multiplication table*:

$*$	r_0	r_1	r_2	s_A	s_B	s_C
r_0	r_0	r_1	r_2	s_A	s_B	s_C
r_1	r_1	r_2	r_0	s_B	s_C	s_A
r_2	r_2	r_0	r_1	s_C	s_A	s_B
s_A	s_A	s_C	s_B	r_0	r_1	r_2
s_B	s_B	s_A	s_C	r_2	r_0	r_1
s_C	s_C	s_A	s_B	r_1	r_2	r_0

Here (for instance) the element s_V at the intersection of the fifth column and the third row is $s_B = r_1 * s_A$, the composition of r_1 and s_A in that order (first the transformation r_1 is performed, then s_A).

As we noted above, composition is *noncommutative*, and this is clearly seen from the table (it is not symmetric with respect to its main diagonal).

The composition operation $*$ in Sym($\triangle$) is (obviously) *associative*, i.e., $(i*j)*k = i*(j*k)$ for all $i, j, k \in$ Sym($\triangle$). The set Sym($\square$) contains the *identity* transformation r_0 (also denoted id or $\mathbf{1}$). Any element i of Sym($\square$) has an *inverse* i^{-1}, i.e., an element such that $i * i^{-1} = i^{-1} * i = \mathbf{1}$.

The set Sym($\triangle$) supplied with the composition operation $*$ is called the *symmetry group of the equilateral triangle*.

1.2.2. Symmetries of the square. Consider all the isometries of the unit square $\square = ABCD$, i.e., all the distance-preserving mappings of the square to itself.

Let us denote by s_H, s_V, and s_{ac}, s_{bd} the reflections in the horizontal and vertical midlines, and in the diagonals AC, BD, respectively. Denote by r_0, r_1, r_2, r_3 the rotations about the center of the

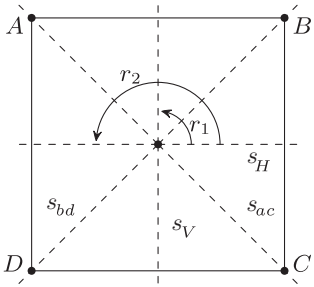


Figure 1.1. Symmetries of the square.

square by 0, 90, 180, 270 degrees, respectively. These eight transformations are all called *symmetries* of the square. We write

$$\mathrm{Sym}(\square) = \{r_0,\, r_1,\, r_2,\, r_3,\, s_H,\, s_V,\, s_{ac},\, s_{bd}\}.$$

Just as in the case of the equilateral triangle, the composition of any two symmetries of the square is a symmetry of the square, and a *multiplication table*, indicating the result of all pairwise compositions, can be drawn up:

$*$	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_0	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_1	r_1	r_2	r_3	r_0	s_{ac}	s_{bd}	s_V	s_H
r_2	r_2	r_3	r_0	r_1	s_V	s_H	s_{bd}	s_{ac}
r_3	r_3	r_0	r_1	r_2	s_{bd}	s_{ac}	s_H	s_V
s_H	s_H	s_{bd}	s_V	s_{ac}	r_0	r_2	r_3	r_1
s_V	s_V	s_{ac}	s_H	s_{bd}	r_2	r_0	r_1	r_3
s_{ac}	s_{ac}	s_H	s_{bd}	s_V	r_1	r_3	r_0	r_2
s_{bd}	s_{bd}	s_V	s_{ac}	s_H	r_3	r_1	r_2	r_0

Here (for instance) the element s_V at the intersection of the sixth column and the fourth row is $s_V = r_2 * s_H$, the composition of r_2 and s_H in that order (first the transformation r_2 is performed, then s_V). Composition is *noncommutative*.

Obviously, composition is *associative*. The set $\mathrm{Sym}(\square)$ contains the *identity* transformation r_0 (also denoted id or **1**). Any element i

of Sym($\square$) has an *inverse* i^{-1}, i.e., an element such that

$$i * i^{-1} = i^{-1} * i = \mathbf{1}.$$

The set Sym($\square$) supplied with the composition operation is called the *symmetry group of the square*.

1.2.3. Symmetries of the cube. Let

$$I^3 = \{(x, y, z) \in \mathbb{R}^3 |\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

be the unit cube. A *symmetry* of the cube is defined as any isometric mapping of I^3 onto itself. The composition of two symmetries (of I^3) is a symmetry. How many are there?

Let us first count the orientation-preserving isometries of the cube (other than the identity), i.e., all its rotations (about an axis) by nonzero angles that take the cube onto itself.

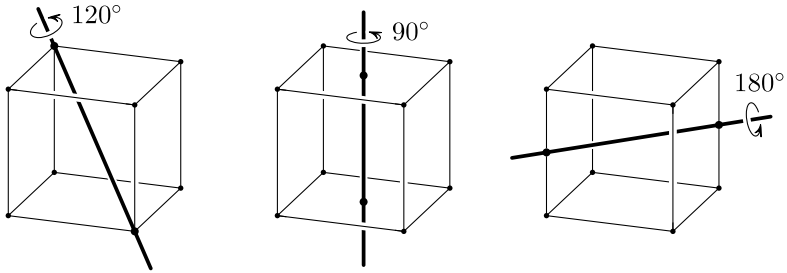


Figure 1.2. Rotations of the cube.

There are three axes of rotation joining the centers of opposite faces, and the rotation angles about each are $\pi/2$, π, $3\pi/2$. There are four axes of rotation joining opposite vertices, the rotation angles for each being $2\pi/3$ and $4\pi/3$. There are six axes of rotation joining midpoints of opposite edges, with only one nonzero rotation for each (by π). This gives us a total of $(3 \times 3) + (4 \times 2) + (6 \times 1) = 23$ orientation-preserving isometries, or 24 if we include the identity.

There are no other orientation-preserving isometries; at this point, we could prove this fact by a tedious elementary geometric counting argument, but we postpone the proof to Chapter 3, where it will be

the immediate result of a more general and sophisticated algebraic method.

There are also 24 orientation-reversing isometries of the cube. Listing them all is the task prescribed by Problem 1.2 (see the end of the chapter), a task which requires little more than a bit of spatial intuition.

Thus the cube has 48 isometries. All their pairwise compositions constitute a multiplication table, which is a 49 by 49 array of symbols, much too unwieldy to fit on a book page.

The set $\mathrm{Sym}(I^3)$ of all 48 symmetries of the cube supplied with the composition operation is called the *symmetry group of the cube*; it is associative, noncommutative, has an identity, and all its elements have inverses, just as the symmetry groups in the two previous examples.

1.2.4. Symmetries of the circle. Let

$$\bigcirc := \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\}$$

be the unit circle. Denote by $\mathrm{Sym}(\bigcirc)$ the set of all its isometries. The elements of $\mathrm{Sym}(\bigcirc)$ are of two types: the rotations r_φ about the origin by angles φ, $\varphi \in [0, 2\pi)$, and the reflections in lines passing through the origin, s_α, $\alpha \in [0, \pi)$, where α denotes the angle from the x-axis to the line (in the counterclockwise direction). The composition of rotations is given by the (obvious) formula

$$r_\phi * r_\psi = r_{(\phi+\psi) \bmod 2\pi},$$

where mod 2π means that we subtract 2π from the sum $\phi + \psi$ if the latter is greater than or equal to 2π.

The composition of two reflections s_α and s_β is a rotation by the angle $|\alpha - \beta|$,

$$s_\alpha * s_\beta = r_{2|\alpha-\beta|}.$$

The interested reader will readily verify this formula by drawing a picture and comparing the angles that will appear when the two reflections are composed.

The set of all isometries of the circle supplied with the composition operation is called the *symmetry group of the circle* and is denoted by $\mathrm{Sym}(\bigcirc)$. The group $\mathrm{Sym}(\bigcirc)$ has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

1.2.5. Symmetries of the sphere. Let

$$\mathbb{S}^2 := \{(x, y) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$$

be the unit sphere. Denote by $\mathrm{Sym}(\mathbb{S}^3)$ the set of all its isometries and by $\mathrm{Rot}(\mathbb{S}^3)$ the set of all its rotations (by different angles about different axes passing through the center of the sphere). Besides rotations, the transformation group $\mathrm{Sym}(\mathbb{S}^3)$ contains reflections in different planes passing through the center of the sphere, its symmetry with respect to its center, and the composition of these transformations with rotations.

Reflections in planes, unlike rotations, reverse the orientation of the sphere. This means that a little circle oriented clockwise on the sphere (if we are looking at it from the outside) is transformed by any reflection into a counterclockwise oriented circle, and the picture of a left hand drawn on the sphere becomes that of a right hand. Now a reflection in a line passing through the sphere's center does not reverse orientation (unlike reflections with respect to a line in the plane!) because a reflection of the sphere in a line is exactly the same transformation as a rotation about this line by 180°. On the other hand, a reflection of the sphere with respect to its center reverses its orientation (again, this is not the case for reflections of the plane with respect to a point).

Note that the composition of two reflections in planes is a rotation (see Problem 1.11), while the composition of two rotations is another rotation (by what angle and about what axis is the question discussed in Problem 1.12).

The set of all isometries of the sphere supplied with the composition operation is called the *symmetry group of the sphere* and is denoted by $\mathrm{Sym}(\mathbb{S}^3)$. The group $\mathrm{Sym}(\mathbb{S}^3)$ has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

1.2.6. A model of elliptic plane geometry. Consider the set $\mathrm{Ant}(\mathbb{S}^2)$ of all pairs of antipodal points (i.e., points symmetric with respect to the origin) on the unit sphere $\mathbb{S}^2$; thus elements of $\mathrm{Ant}(\mathbb{S}^2)$ are *not* ordinary points, but *pairs of points*. Now consider the family (that we denote $\mathrm{O}(3)$) of all isometries of the space $\mathbb{R}^3$ that do not move the origin.[1] Clearly, any such isometry takes pairs of antipodal points to pairs of antipodal points, thus it maps the set $X = \mathrm{Ant}(\mathbb{S}^2)$ to itself.

The family $\mathrm{O}(3)$ of transformations of the set $\mathrm{Ant}(\mathbb{S}^2)$ is called the *isometry group of the Riemannian elliptic plane*. This is a much more complicated object than the previous "toy geometries". We will come back to its study in Chapter 6.

1.3. Transformation groups

1.3.1. Definitions and notation. Let X be a set (finite or infinite) of arbitrary elements called *points*. By definition, a *transformation group G acting on X* is a (nonempty) set G of bijections of X supplied with the composition operation $*$ and satisfying the following conditions:

(i) G is closed under composition, i.e., for any transformations $g, g' \in G$, the composition $g * g'$ belongs to G;

(ii) G is closed under taking inverses, i.e., for any transformation $g \in G$, its inverse g^{-1} belongs to G.

These conditions immediately imply that G contains the identity transformation. Indeed, take any $g \in G$; by (ii), we have $g^{-1} \in G$; by (i), we have $g^{-1} * g \in G$; but $g^{-1} * g = \mathrm{id}$ (by definition of inverse element), and so $\mathrm{id} \in G$. Note also that composition in G is associative (because the composition of mappings is always associative).

If $x \in X$ and $g \in G$, then by xg we denote the image of the point x under the transformation g. (The more usual notation $g(x)$ is not convenient: we have $x(g*h) = (xg)h$, but $(g*h)(x) = h(g(x))$, with g and h appearing in reverse order in the right-hand side of this equality.)

[1] In linear algebra courses such transformations are called *orthogonal* and $\mathrm{O}(3)$ is called the *orthogonal group*.

1.3.2. Examples. The five toy geometries considered in the previous section all give examples of transformation groups. The five transformation groups Sym act (by isometries) on the equilateral triangle, the square, the cube, the circle, and the sphere, respectively. In the last example (1.2.5), the orthogonal group O(3) acts on pairs of antipodal points on the sphere, these pairs being regarded as "points" of the "elliptic plane".

More examples are given by the transformation group consisting of all the bijections $\mathrm{Bij}(X)$ of any set X. By definition of transformation groups, $\mathrm{Bij}(X)$ is the largest (by inclusion) transformation group acting on the given set X. At the other extreme, any set X has a transformation group consisting of a single element, the identity transformation.

When the set X is finite and consists of n objects, the group $\mathrm{Bij}(X)$ of all its bijections is called the *permutation group* on n objects and is denoted by S_n. This is one of the most fundamental notions of mathematics, and plays a key role in abstract algebra, linear algebra, and, as we shall see already in the next chapter, in geometry.

1.3.3. Orbits, stabilizers, class formula. Let $(X : G)$ be some transformation group acting on a set X and let $x \in X$. Then the *orbit* of x is defined as

$$\mathrm{Orb}(x) := \{xg \mid g \in G\} \subset X,$$

and the *stabilizer* of x is

$$\mathrm{St}(x) := \{g \in G \mid xg = x\} \subset G.$$

For example, if $X = \mathbb{R}^2$ and G is the rotation group of the plane about the origin, then the set of orbits consists of the origin and all concentric circles centered at the origin; the stabilizer of the origin is the whole group G, and the stabilizers of all the other points of $\mathbb{R}^2$ are trivial (i.e., they consist of one element – the identity $\mathrm{id} \in G$).

Suppose $(X : G)$ is an action of a finite transformation group G on a finite set X. Then the number of points of G is (obviously) given by

$$\boxed{|G| = |\mathrm{Orb}(x)| \times |\mathrm{St}(x)|} \tag{1.1}$$

for any $x \in X$. Now let $A \subset X$ be a set that intersects each orbit at exactly one point. Then the number of points of X is given by the formula

$$\boxed{|X| \;=\; \sum_{x \in A} \frac{|G|}{|\mathrm{St}(x)|}}, \tag{1.2}$$

called the *class formula*. This formula, just as the previous one, follows immediately from definitions.

1.3.4. Fundamental domains. If X is a subset of $\mathbb{R}^n$ (e.g. $\mathbb{R}^n$ itself) and G is a transformation group acting on X, then a subset $F \subset X$ is called a *fundamental domain* of the action of G on X if

- F is an open set in X;
- $F \cap Fg = \varnothing$ for any $g \in G$ (except $g = \mathrm{id}$);
- $X = \bigcup_{g \in G} \mathrm{Clos}(Fg)$, where Clos(.) denotes the closure of a set.

For example, in the case of the square, a fundamental domain of the action of $\mathrm{Sym}(\square)$ is the interior of the triangle AOM, where O is the center of the square and M is the midpoint of side AB; of course $\mathrm{Sym}(\square)$ has many other fundamental domains. Thus fundamental domains are not necessarily unique. Moreover, fundamental domains don't always exist: for instance, $\mathrm{Sym}(S^1)$ (and other "continuous" geometries) does not have any fundamental domains.

1.3.5. Morphisms. According to one of the main principles of the category approach to mathematics, as soon as an important class of objects is defined, one must define their *morphisms*, i.e., the natural class of relationships between them. Following this principle, we say that a mapping of transformation groups $\alpha : G \to H$ is a *homomorphism* if α respects the product (composition) structure, i.e.,

$$\alpha(g_1 * g_2) = \alpha(g_1) * \alpha(g_2) \quad \text{for all} \quad g_1, g_2 \in G. \tag{1.3}$$

Let us look at a few examples of homomorphisms:

(i) the mapping $\mu : \mathrm{Sym}(\square) \to \mathrm{Sym}(I^3)$ obtained by placing the square on top of the cube and extending its isometries to the whole cube in the natural way (e.g. assigning the rotation by 90° about the vertical axis passing through the centers of the horizontal faces of the cube to the 90° rotation of the square);

(ii) the mapping $\nu : \mathrm{Sym}(\triangle) \to \mathrm{Sym}(\bigcirc)$ assigning to each rotation of the triangle the rotation of the circle by the same angle and, to the reflections s_A, s_B, s_C, the reflections $s_0, s_{2\pi/3}, s_{4\pi/3}$ of the circle;

(iii) the mapping $\pi : \mathrm{Rot}(I^3) \to \mathrm{Rot}(\square)$ induced by the projection of the cube on its bottom horizontal face Φ, i.e., assigning the identity element to all isometries of the cube that do not map Φ to Φ, and assigning, to all the other isometries of the cube, their restriction to Φ;

(iv) the mapping $\iota : S_3 \to \mathrm{Sym}(\triangle)$ assigning to each permutation of the symbols A, B, C the isometry that performs that permutation of the vertices A, B, C of the triangle.

The proof of the fact that these mappings are indeed homomorphisms, i.e., relation (1.3) holds, is a straightforward verification left to the reader.

A homomorphism α of transformation groups is said to be a *monomorphism* if the mapping α is injective (i.e., takes different elements to different ones). Examples of monomorphisms are the homomorphisms μ and ν above. A homomorphism α of transformation groups is said to be an *epimorphism* if α is surjective (i.e., is an onto map). An example is the mapping π above. A homomorphism α of transformation groups is said to be an *isomorphism* if it is both a monomorphism and an epimorphism, i.e., if the mapping α is bijective.

Two transformation groups G and H acting on two sets X and Y (the case $X = Y$ is not forbidden) are called *isomorphic* if there exists an isomorphism $\phi : G \to H$. If two isomorphic groups are finite, then they necessarily have the same number of elements (but the number of points in the sets on which they act can differ, as for example in the

case of the isomorphic groups Sym(△) and S_3). Note in this context that Sym(□) and S_4 are *not* isomorphic, because the first of these groups consists of 8 elements, while the second has $4! = 24$.

1.3.6. Order. The *order* of a transformation group G is, by definition, the number of its elements; we denote it by $|G|$. Thus

$$|\mathrm{Sym}(\triangle)| = 6, \quad |\mathrm{Sym}(\square)| = 8, \quad |\mathrm{Sym}(\bigcirc)| = \infty.$$

The *order* of an element g of a transformation group G is, by definition, the least positive integer k such that the element $g*g*\cdots*g$ (k factors) is the identity; this integer is denoted by $\mathrm{ord}(g)$; if there is no such integer, then g is said to be of *infinite order*. For example, the rotation by 30° in Sym(◯) has order 12, while the rotation by $\sqrt{2}\pi$ is of infinite order. (The last fact follows from the irrationality of $\sqrt{2}$.)

1.3.7. Subgroups. Many important classes of objects have naturally defined "subobjects" (e.g. spaces and subspaces, manifolds and submanifolds, algebras and subalgebras). Transformation groups are no exception: if G is a transformation group and H is a subset of G, then H is called a *subgroup* of G if H itself is a group with respect to the composition operation $*$, i.e., if it satisfies the two conditions:

(i) H is closed under composition, i.e., $g, g' \in H \Longrightarrow g * g' \in H$;

(ii) H is closed under taking inverses, i.e., $g \in G \Longrightarrow g^{-1} \in G$.

According to this definition, any transformation group G has at least two subgroups: G itself and its one-element subgroup, i.e., the group {id} consisting of the identity element. We will call these two subgroups *trivial*, and all the others, *nontrivial*.

For example, the subset of all rotations of the group Sym(□) is a (nontrivial) subgroup of Sym(□) (of order 4), the set consisting of the identity element and a reflection s_α is a subgroup of order 2 in Sym(◯), while the set of all rotations of Sym(◯) is a subgroup of infinite order.

If g is an element of order k in a transformation group G, then the set of k elements$\{g,\ g*g, \ldots,\ g*g*\cdots*g = \mathrm{id}\}$ is a subgroup of G of order k; it is called the *cyclic subgroup of G generated by g*.

This terminology is also used when g is an element of infinite order, but then the subgroup $\{\mathrm{id},\, g,\, g*g, \ldots,\, g*g*\cdots*g, \ldots\}$ is also of infinite order.

1.4. The category of geometries

In this section, we present the main definition of this course (that of a geometry) and define some related basic concepts.

1.4.1. Geometries in the sense of Klein. A pair $(X : G)$, where X is a set and G is a transformation group acting on X will be called a *geometry in the sense of Klein*. The six examples in Sec. 1.2 define the geometry of the equilateral triangle, the geometry of the square, the geometry of the cube, the geometry of the circle, the geometry of the sphere, and the geometry of Riemann's elliptic plane. Another example is the set $\mathrm{Bij}(X)$ of all bijections of any set X.

1.4.2. The Erlangen program. The idea that geometries are sets of objects with transformation groups acting on them was first stated by the German mathematician Felix Klein in 1872 in a famous lecture at Erlangen. In that lecture (for an English translation, see [10]), he enunciated his views on geometries in the framework of what became known as the "Erlangen program".

There is no doubt that all the geometries known in the times of Klein satisfy the property that he gave in his lecture, and so do all the geometries that were developed since then. However, this property can hardly be said to *characterize* geometries: it is much too broad. Thus, in the sense of the formal definition from the previous subsection, the permutation group is a geometry, and so is any topological space, any abstract group, even any set.

Nevertheless, we will stick to the notion of geometry given in 1.4.1 for want of a more precise formal definition. Such a definition, if it existed, would require supplying $(X : G)$ with additional structures (besides the action of G on X), but it is unclear at this time what these structures ought to be. If one looks at such branches of mathematics as global differential geometry, geometric topology, and differential

topology, there appears to be no consensus among the experts about where geometry ends and topology begins in those fields.

The definition in 1.4.1 may be too broad, but it has the advantage of being extremely succinct and leading to the definition of a very natural category.

1.4.3. Morphisms. According to the general philosophy underlying the category language, a morphism from one geometry to another should be defined as a mapping of the set of points of one geometry to the set of points of the other that respects the actions of the corresponding transformation groups. More precisely, given two geometries $(G : X)$ and $(H : Y)$, a *morphism* (or an *equivariant map*) is any pair (α, f) consisting of a homomorphism of transformation groups $\alpha : G \to H$ and a mapping of sets $f : X \to Y$ such that

$$\boxed{f(xg) = (f(x))(\alpha(g))} \tag{1.4}$$

for all $x \in X$ and all $g \in G$. This definition is typical of the category approach in mathematics: at first glance, the boxed formula makes no sense at all (no wonder category theory is called abstract nonsense), but actually the definition is perfectly natural.

To see this, let us take any point $x \in X$ and let an arbitrary transformation $g \in G$ act on x, taking it to the point $xg \in X$. Under the map $f : X \to Y$, the point x is taken to the point $f(x) \in Y$ and the point xg is taken to the point $f(xg) \in Y$. How are these two points related? What transformation (if any) takes $f(x)$ to $f(xg)$? Clearly, if the pair of maps (f, α) respects the action of the transformation groups in X and Y, it must be none other than $\alpha(g)$, and this is precisely what the boxed formula says.

To check that the reader has really understood this definition, we suggest that she/he prove that $\alpha(\mathbf{1})=\mathbf{1}$ for any morphism (f, α).

1.4.4. Isomorphic geometries. In any mathematical theory, isomorphic objects are those which are equivalent, i.e., are not distinguished in the theory. Thus isomorphic linear spaces are not distinguished in linear algebra, sets of the same cardinality (i.e., sets for which there exists a bijective map) are equivalent in set theory,

isomorphic fields are not distinguished in abstract algebra, congruent triangles are the same in Euclidean plane geometry, and so on. What geometries should be considered equivalent? We hope that the following definition will seem natural to the reader.

Two geometries $(X : G)$ and $(Y : H)$ are called *isomorphic*, if there exist a bijection $f : X \to Y$ and an isomorphism $\alpha : G \to H$ such that

$$f(xg) = (f(x))(\alpha(g)) \quad \text{for all} \quad x \in X \quad \text{and all} \quad g \in G.$$

In the definition, the displayed formula is a repetition of relation (1.4), so it expresses the requirement that an isomorphism be a morphism (must satisfy the equivariance condition, i.e., respect the action of the transformation groups), the conditions on α and f say that they are equivalences, so what this definition is saying is that $(X : G)$ and $(Y : H)$ are the same.

At this stage we have no meaningful examples of isomorphic geometries. They will abound in what follows. For instance, we will see (in Chapter 10) that the Poincaré half-plane model of hyperbolic geometry is isomorphic to the Cayley–Klein disk model.

1.4.5. Subgeometries. What are subobjects in the category of geometries? The reader who is acquiring a feel for the category language should have no difficulty in coming up with the following definition. A geometry $(G : X)$ is said to be a *subgeometry* of the geometry $(H : Y)$ if X is a subset of Y, G is a subgroup of H, and the pair $(\mathrm{id}_X, \mathrm{id}_G)$, where $\mathrm{id}_G : g \mapsto g \in H$ and $\mathrm{id}_X : x \mapsto x \in Y$ are the identities, is a morphism of geometries.

A closely related definition is the following. An *embedding* (or *injective morphism*) of the geometry $(X : G)$ to the geometry $(Y : H)$ is a morphism (f, α) such that $\alpha : G \to H$ is a monomorphism and $f : X \to Y$ is injective.

Examples of subgeometries and embeddings of geometries can easily be deduced from the examples of subgroups of transformation groups in Subsection 1.3.4.

1.5. Some philosophical remarks

The examples in Section 1.2 (square, cube, circle) were taken from elementary school geometry. This was done *to motivate* the choice of the action of the corresponding transformation group. But now, in the example of the cube, let us forget school geometry: instead of the cube I^3 with its vertices, edges, faces, angles, interior points and other structure, consider the abstract set of points $\{A, B, C, D, A', B', C', D'\}$ and define the "isometries" of this "cube" as a set of 48 bijections; for example, the "rotation by 270 degrees" about the vertical axis is the bijection

$$A \mapsto B,\ B \mapsto C,\ C \mapsto D,\ D \mapsto A,$$
$$A' \mapsto B',\ B' \mapsto C',\ D' \mapsto D',\ D' \mapsto A',$$

and the 47 other "isometries" are defined similarly. Then (still forgetting school geometry), we can *define* vertices, edges (AB is an edge, but AC' is not), faces, prove that all edges are congruent, all faces are congruent, the "cube" can "rotate" about each vertex, etc.). The result is the *intrinsic geometry of the set of vertices* of the cube.

This geometry is not the same as the geometry (I^3: $\mathrm{Sym}(I^3)$) of the cube described in Subsection 1.2.3. Of course the group G acting in these two geometries is the *same group* of order 48, but it acts on two *different sets*: the (infinite) set of points of the cube I^3 and the (finite) set of its 8 vertices $A, B, C, D, A', B', C', D'$. Thus the algebra of the two situations is the same, but the geometry is different. The geometry of the solid cube I^3 is of course much richer than the geometry of the vertex set of the cube. For example, we can define line segments inside the cube, establish their congruence, etc.

Note also that the geometric properties of the cube I^3 *regarded as a subset of Euclidean space* $\mathbb{R}^3$ are richer than its properties coming from its own geometry (I^3 : $\mathrm{Sym}(I^3)$), e.g., segments of the same length inside the cube, which are always congruent in the geometry of $\mathbb{R}^3$, don't have to be congruent in the geometry of the cube!

Another example: the set of three points $\{A, B, C\}$ with two transformations, namely the identity and the "reflection"

$$A \mapsto A,\ B \mapsto C,\ C \mapsto B,$$

is of course a geometry in the sense of Klein. What should it be called? An appropriate title, as the reader will no doubt agree, is "the intrinsic geometry of the vertex set of the isosceles triangle".

1.6. Problems

1.1. List all the elements (indicating their orders) of the symmetry group (i.e., isometry group) of the equilateral triangle. List all its subgroups. How many elements are there in the group of motions (i.e., orientation-preserving isometries) of the equilateral triangle?

1.2. Answer the same questions as in Problem 1.1 for

(a) the regular pyramid with four lateral faces;

(b) the regular tetrahedron;

(c) the cube;

(d)* the dodecahedron; [2]

(e)* the icosahedron;

(f) the regular n-gon (i.e., the regular polygon of n sides); consider the cases of odd and even n separately.

1.3. Embed the geometry of the motion group of the square into the geometry of the motion group of the cube, and the geometry of the circle into the geometry of the sphere.

1.4. For what values of n and m can the geometry of the regular n-gon be embedded in the geometry of the regular m-gon?

1.5. Let G be the symmetry group of the regular tetrahedron. Find all its subgroups of order 2 and describe their action geometrically.

1.6. Let G^+ be the group of motions of the cube. Indicate four subsets of the cube on which G^+ acts by all possible permutations.

1.7. Find a minimal system of generators (i.e., a set of elements such that any element can be represented as the product of some elements from this set) for the symmetry group of

(a) the regular tetrahedron;

(b) the cube.

[2] Here and in what follows the asterisk after a problem number means that the problem is not easy and should be regarded as a challenge.

1.8. Describe the fundamental domains of the symmetry group of

(a) the cube;

(b) the icosahedron;

(c) the regular tetrahedron.

1.9. Describe the Möbius band as a subset of $\mathbb{R}P^2$.

1.10. Show that the composition of two reflections of the sphere in planes passing through its center is a rotation. Determine the axis of rotation and, if the angle between the planes is given, the angle of rotation.

1.11. Given two rotations of the sphere, describe their composition.

Chapter 2

Abstract Groups and Group Presentations

In order to study geometries more complicated than the toy models with which we played in the previous chapter, we need to know much more about group theory. In this chapter we present the relevant facts of this theory (they will constantly be used in what follows).

The theory of transformation groups began in the work of several great mathematicians: Lagrange, Abel, Galois, Sophus Lie, Felix Klein, Élie Cartan, Hermann Weyl. At the beginning of the 20th century, algebraists decided to generalize this theory to the formal theory of *abstract groups*. In this chapter, we will study this formal theory and learn that it is not a generalization at all: Cayley's Theorem (which concludes this chapter) says that all abstract groups are actually transformation groups. We will also learn that two important classes of groups (*free groups* and *permutation groups*) have certain universality properties. Finally, we will learn about *group presentations*, which allow us to replace computations in groups by games with words.

2.1. Abstract groups

2.1.1. Groups: definition and manipulation. By definition, an (abstract) *group* is a set G of arbitrary elements supplied with a binary

operation $*$ (usually called *multiplication*) if it obeys the following rules:

- (*neutral element axiom*) there exists a unique element $e \in G$ called *neutral* such that $g * e = e * g = g$ for any $g \in G$;
- (*inverse element axiom*) for any element $g \in G$ there exists a unique element in G, denoted g^{-1} and called *inverse to* g, such that $g * g^{-1} = g^{-1} * g = e$;
- (*associativity axiom*) $(g*h)*k = g*(h*k)$ for all $g, h, k \in G$.

A group $(G, *)$ is called *commutative* or *Abelian* if $g * h = h * g$ for all $g, h \in G$ (in that case the operation is usually called a *sum* and the inverse element is usually denoted by $-g$ instead of g^{-1}).

Note that the elements of an abstract group can be objects of any nature, they are not necessarily bijections of something and the operation $*$ is not necessarily composition, while the notation g^{-1} for inverse elements is purely formal, it does not mean that g^{-1} is the inverse of a bijection.

The three axioms for groups listed above are much stronger than necessary. For example, the uniqueness condition in the inverse element axiom can be omitted without changing the class of objects defined by these axioms. The definition can be weakened further, but this is not an important fact from the point of view of geometry, so we do not dwell on it further.

The group axioms have some obvious consequences that are useful when performing calculations with elements of groups. In these calculations and further on, we omit the group operation symbol, i.e., we write gh instead of $g * h$.

The first immediate consequence of the group axioms are the *left* and *right cancellation rules*, which say that one can cancel equal terms on the two sides of an equation, provided they both appear at the left (or at the right) of the corresponding expression, i.e.,

$$\forall g, h, k \in G \quad gh = gk \iff h = k, \quad hg = kg \iff h = k.$$

The implications in these formulas are two-sided; reading them from right to left, we can say that one can multiply both sides of an equation by the same element *from the same side*. The phrase in italics is

of course important, because for non-Abelian groups the cancellation of equal terms on different sides of an equation can result in a false statement.

Another simple but important consequence of the axioms is the *rule for solving linear equations*, i.e.,

$$\forall g, h, x \in G \quad gx = h \iff x = g^{-1}h, \quad xg = h \iff x = hg^{-1},$$

which are proved by multiplying both sides by the element g^{-1} (it exists by the inverse element axiom) from the left and the right, respectively, using associativity and the neutral element axiom.

These two rules are constantly used when performing manipulations with equations in groups, as the reader will see in solving some of the exercises at the end of this chapter.

2.1.2. Examples of groups. It is easy to see that any transformation group is a group. Indeed, the three axioms of abstract groups listed above, although they do not appear explicitly in the definition of transformation groups, hold automatically for the latter, because their elements are not arbitrary objects, they are bijections, and the multiplication operation is not arbitrary (it is composition): for them associativity and the neutral element axiom hold automatically.

Here are some other important examples of groups.

(i) *The standard numerical groups*: the integers under addition $(\mathbb{Z}, +)$, as well as the rational, real, and complex numbers under addition $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$; the nonzero rational, real, and complex numbers under multiplication $(\mathbb{Q} \setminus \{0\}, \times)$, $(\mathbb{R} \setminus \{0\}, \times)$, and $(\mathbb{C} \setminus \{0\}, \times)$. Note that the nonzero integers under multiplication are *not* a group (no inverse elements!), neither are the natural numbers $\mathbb{N}$ under addition (for the same reason). Another nice numerical group is formed by the *unimodular complex numbers* under multiplication $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$.

(ii) *The group of residues modulo* m, $(Z_m, \oplus)$ (also known as the m-element *cyclic group*); its elements are the m infinite sets of integers that have the same remainder under division by the natural number

m; we denote these sets by $\overline{0}, \overline{1}, \dots, \overline{m-1}$; their sum $\oplus$ is defined by

$$\overline{i} \oplus \overline{j} := \overline{(i+j) \bmod m},$$

where $(\cdot) \bmod m$ stands for the remainder under division by m. The sum operation $\oplus$ is well defined, i.e., does not depend on the choice of the representatives i and j in the classes $\overline{i}$ and $\overline{j}$. Indeed, if we take $i + rm$ instead of i and $j + sm$ instead of j, then

$$\overline{(i+rm)+(j+sm)} = \overline{(i+j+(r+s)m)} = \overline{(i+j)}.$$

(iii) *The group of permutations of n objects S_n*: its elements are bijections of a set of n elements that we denote by natural numbers $(\{1, 2, \dots n\})$; we will write bijections $s \in S_n$ in the form

$$s = [i_1, i_2, \dots, i_n], \quad \text{where} \quad i_1 = s(1),\ i_2 = s(2), \dots, i_n = s(n);$$

multiplication in S_n is the composition of bijections. This group is extremely important not only in geometry, but also in linear algebra, combinatorics, representation theory, mathematical physics, etc. We will come back to permutation groups later in this chapter.

(iv) *The free group $\mathbb{F}_n = \mathbb{F}(a_1, \dots, a_n)$ on n generators*; its elements are equivalence classes of words and the group operation is concatenation; a detailed definition of $\mathbb{F}_n$ appears in Subsection 2.6.2 below.

(v) *The group* $\mathrm{GL}(n)$ *of nonsingular linear operators on* $\mathbb{R}^n$; its elements are n by n matrices with nonzero determinant and the group operation is matrix multiplication (or, which is the same thing, composition of operators).

(vi) *The groups of orthogonal and special orthogonal operators on* $\mathbb{R}^n$, standardly denoted by $\mathrm{O}(n)$ and $\mathrm{SO}(n)$. We assume that the reader is familiar with the groups $\mathrm{GL}(n)$, $\mathrm{O}(n)$, and $\mathrm{SO}(n)$ at least for $n = 2$ and $n = 3$; if this is not the case, he/she is referred to any introductory linear algebra course. In what follows, we will only need the low-dimensional case ($n = 2, 3$), and when we do, the corresponding definitions will be given.

2.1.3. Order of a group and of its elements, generators. The notions of *order* (of elements of a group and of the group itself) and of *generator* for abstract groups are defined exactly as for transformation groups (see Section 1.3). In this book, $|G|$ denotes the *order of the group* G (i.e., the number of its elements), $\operatorname{ord}(g)$ denotes the *order of the element* $g \in G$, i.e., the least natural number k such that $g^k = e$. For example: $|\mathbb{Z}_5| = 5$; $|\mathrm{Sym}(\bigcirc)| = \infty$; for $\overline{3} \in \mathbb{Z}_{15}$, we have $\operatorname{ord}(\overline{3}) = 5$; for any nonzero real number x in the additive group $\mathbb{R}$, we have $\operatorname{ord}(x) = \infty$.

A family of *generators* of a group G is a (finite or infinite) set of its elements $\{g_1, g_2, \dots\}$ in terms of which any element g of G can be expressed, i.e., written in the form $g = g_1^{\varepsilon_1} \dots g_k^{\varepsilon_k}$, where the ε_i's equal ± 1 and g_i^{+1} stands for g_i. For example, any nonzero element of $\mathbb{Z}_p$, where p is prime, constitutes a (one-element) family of generators for $\mathbb{Z}_p$, while $\mathrm{Sym}(\bigcirc)$ does not have a finite family of generators. If g is an element of order m of a group G, then the set $\{g, g^2, \dots, g^{m-1}, g^m = e\}$ is also a group (it is a "subgroup" of G, see the definition in 2.3.1), and its order is m. This justifies the use of the same term "order" for groups and their elements, i.e., for notions that seem very different at first glance.

2.2. Morphisms of Groups

In accordance with the traditions of the category language, as soon as we have defined an interesting class of objects, in this case groups, we should define their morphisms.

2.2.1. Definitions. Suppose $(G, *)$ and $(H, \star)$ are groups; a mapping $\phi : G \to H$ is called a *homomorphism* (or a *morphism of groups*) if it respects the operations, i.e.,

$$\varphi(g_1 * g_2) = \varphi(g_1) \star \phi(g_2).$$

Thus, the inclusion $\mathbb{Z} \to \mathbb{R}$, $n \mapsto n$, is a morphism, while the inclusion $(\mathbb{Q} \setminus \{0\}, \times) \to (\mathbb{Q}, +)$ is not (the operations are not respected, e.g., $2 \times 3 \neq 2 + 3$).

By definition, a homomorphism φ is a *monomorphism* (respectively, an *epimorphism* or an *isomorphism*) if the mapping φ is injective (resp., surjective or bijective). From the point of view of abstract algebra, isomorphic groups are identical.

2.2.2. Examples. The group $\mathrm{Sym}(\triangle)$ of isometries of the equilateral triangle is isomorphic to the permutation group S_3, the group $\mathrm{Sym}(\bigcirc)$ is isomorphic to $\mathrm{SO}(2)$, the group of motions of the Euclidean plane $\mathbb{R}^2$ preserving the origin; there are obvious monomorphisms of the rotation group $\mathrm{Rot}(\square)$ into $\mathrm{SO}(2)$ and of $\mathbb{Z}_3$ into $\mathbb{Z}_{15}$; there is an equally obvious epimorphism of $\mathbb{Z}$ onto $\mathbb{Z}_{17}$.

2.3. Subgroups

Worthwhile mathematical objects should not only be related by morphisms, they should have naturally defined subobjects.

2.3.1. Definitions and examples. A *subgroup* H of a group G is a subset of G which satisfies the group axioms. Note that in order to check that H is a subgroup of G, it is not necessary to verify all the group axioms; it suffices to check that H is closed under the group operation and under taking inverses. Any group G has at least two subgroups: the one-element subgroup consisting of the neutral element $e \in G$ and the group G itself. These two subgroups are sometimes called *trivial*, and of course in the study of the structure of groups we are interested in *nontrivial* subgroups.

Examples: $\mathrm{Rot}(\bigcirc)$ is a subgroup of the group $\mathrm{Sym}(\bigcirc)$, the set $\{[1234], [2134]\}$ is a subgroup of S_n, the set $\{\overline{0}, \overline{5}, \overline{10}\}$ is a subgroup of $\mathbb{Z}_{15}$, while $\{\overline{0}, \overline{5}, \overline{11}\}$ is not.

2.3.2. Partition of a group into cosets. If H is a subgroup of G, then the (*left*) *coset* $gH \subset G$, for some $g \in G$, is the set of all elements of the form gh for $h \in H$. Right cosets Hg are defined similarly. *Right cosets as well as left cosets form a partition of the set of elements of a group*, i.e., two cosets either do not intersect or coincide.

To prove this, it suffices to show that if two cosets have a common element $\overline{g} \in g_1H \cap g_2H$, then any element of g_1H belongs to g_2H and vice versa. So suppose that $\widetilde{g} \in g_1H$ (which means that $\widetilde{g} = g_1\widetilde{h}$

for some $\widetilde{h} \in H$); we must show that $\widetilde{g} \in g_2H$, i.e., we must find an $h_x \in H$ such that $\widetilde{g} = g_2h_x$.

Since $\overline{g} \in g_1H \cap g_2H$, there exist elements $\overline{h}_1, \overline{h}_2 \in H$ for which we have $g_1\overline{h}_1 = \overline{g} = g_2\overline{h}_2$, which implies that $g_1 = g_2\overline{h}_2(\overline{h}_1)^{-1}$. Now we can write

$$\widetilde{g} = g_1\widetilde{h} = g_2\overline{h}_2(\overline{h}_1)^{-1}\widetilde{h} = g_2\left(\overline{h}_2(\overline{h}_1)^{-1}\widetilde{h}\right) = g_2h_x,$$

where we have defined h_x as $\overline{h}_2(\overline{h}_1)^{-1}\widetilde{h}$, and since h_x belongs to H (as the product of elements of H), we have proved the implication $\widetilde{g} \in g_1H \Longrightarrow \widetilde{g} \in g_2H$. The reverse implication is proved by a symmetric argument (interchange the indices 1 and 2).

Thus we have obtained the partition of G into left cosets. The partition into right cosets is obtained similarly.

Note also that all cosets have the same number of elements (finite or infinite), because there is an obvious bijection between any coset and the subgroup H. This bijection for left cosets is given by the rule $gH \ni gh \mapsto h \in H$.

2.4. The Lagrange theorem

The corollary to the elementary theorem proved below is the first structure theorem about abstract groups. It was proved (for transformation groups) almost two centuries ago by Lagrange.

Theorem 2.4.1. *If H is a subgroup of a finite group G, then the order of H divides the order of G.*

Proof. The cosets of H in G form a partition of the set of elements of G (see 2.3.2) and all have the same number of elements as H. □

Corollary 2.4.2. *Any group G of prime order p is isomorphic to $\mathbb{Z}_p$.*

Proof. Let $g \in G$, $g \neq e$. Let m be the smallest positive integer such that $g^m = e$. Then it is easy to see that $H := \{e, g, g^2, \ldots, g^{m-1}\}$ is a subgroup of G. By Theorem 2.4.1, m divides p. This is impossible unless $m = p$, but then $H = G$ is obviously isomorphic to $\mathbb{Z}_p$. □

2.5. Quotient groups

Nice mathematical objects often have naturally defined "quotient objects" obtained by "dividing out" the given object by some subobject (examples that may be familiar to the reader are quotient spaces in linear algebra). The construction of "quotient groups" along those lines works only when the subgroup used is in a sense "nice", and we begin by defining such subgroups.

2.5.1. Normal subgroups. A subgroup $H \subset G$ is *normal* if we have $gHg^{-1} = H$ for any $g \in G$, i.e., for any $h \in H$ and any $g \in G$ the inclusion $g^{-1}hg \in H$ holds.

An example of a normal subgroup is the set $\{\overline{0}, \overline{5}, \overline{10}\}$ in $\mathbb{Z}_{15}$. More generally, any subgroup of an Abelian group is (obviously!) normal.

To see an example of a subgroup which is *not* normal, consider the subset $D := \{e = [1,2,3,4], [2,1,3,4]\}$ in the permutation group S_4. The set D is obviously a subgroup (isomorphic to $\mathbb{Z}_2$) of S_4, but it is not normal, because

$$[4,1,2,3]\,[2,1,3,4]\,[2,3,4,1] = [1,3,2,4] \notin D.$$

2.5.2. Construction of quotient groups. If H is a normal subgroup of G, there is a well-defined operation in the family of cosets: the product of two cosets is the coset containing the product of any two elements of these cosets. For left cosets this may be written as $(g_1)H\,(g_2)H := (g_1g_2)H$.

To prove that this is an operation well-defined on cosets, we must show that if we replace g_1 by another element $\overline{g}_1$ from Hg_1 and replace g_2 by another element $\overline{g}_2$ from Hg_2, then $\overline{g}_1H\,\overline{g}_2H = g_1H\,g_2H$. Without loss of generality, it suffices to consider the case in which only one of the two elements is replaced, say g_1. Then we have $\overline{g}_1 = g_1\overline{h}_1$ for some $\overline{h}_1 \in H$. We must prove that $\overline{g}_1g_2 \in g_1g_2H$, i.e., that there exists an h_x such that $\overline{g}_1g_2 = g_1g_2h_x$. Replacing $\overline{g}_1$ by its expression $g_1\overline{h}_1$ (see above), we can rewrite the previous equation as

$$g_1\overline{h}_1g_2 = g_1g_2h_x.$$

Solving this (linear) equation for h_x, we obtain $h_x = g_2^{-1}\overline{h}_1g_2$. Recall that H is normal, therefore the right-hand side of the previous

equality is an element of H. Thus we have found the required element $h_x \in H$, thereby proving that the product of cosets is well defined.

The family of cosets supplied with this product operation is called the *quotient group* of G by H and is denoted by G/H. It is easy to show that G/H satisfies the axioms for groups.

Example: in the additive group of integers $(\mathbb{Z}, +)$, elements of the form $5k$, $k \in \mathbb{Z}$, constitute a normal subgroup (of infinite order), denoted $5\mathbb{Z}$; the corresponding quotient group $\mathbb{Z}/5\mathbb{Z}$ is isomorphic to the group $\mathbb{Z}_5$.

2.6. Free groups and permutations

In this section, we study two classes of groups: the free groups (which have the "least structure") and the permutation groups (which have the "most structure").

2.6.1. Free groups. Let $\{a_1, \dots, a_k\}$ be a set of symbols. Then the set of formal symbols (called *letters*)

$$A := \{e, a_1, \dots, a_k, a_1^{-1}, \dots, a_k^{-1}\}$$

will be our *alphabet*. A string of letters from our alphabet will be called a *word*. Two words w_1 and w_2 are called *equivalent*, if one can be obtained from the other by using the following *trivial relations*: $a_i a_i^{-1} = a_i^{-1} a_i = e$ for any i and $ae = ea = a$ for any $a \in A$; for example,

$$a_1 a_3^{-1} \sim a_1 a_3^{-1} e \sim a_1 a_3^{-1} a_2 a_2^{-1} \sim a_1 a_3^{-1} a_2 e a_2^{-1}.$$

The *product* of two words is defined as their concatenation (i.e., the result of writing them one after the other). The *free group* with generators $a_1, \dots, a_k$ is defined as the set of equivalence classes of words supplied with the product (concatenation) operation and is denoted by $\mathbb{F}_k = \mathbb{F}[a_1, \dots, a_k]$. The fact that concatenation is well defined on equivalence classes (i.e., the concatenation of equivalent elements produces an element from the same equivalence class) is obtained by an straightforward verification, which we omit.

For example, $\mathbb{F}[a]$ is isomorphic to $(\mathbb{Z}, +)$, while $\mathbb{F}[a_1, a_2]$ is not commutative.

2.6.2. Permutation groups. The *permutation group* S_n on n objects was defined in 2.1.2 as the family of all bijections of the set $\{1, 2, \dots, n\}$ supplied with the operation of composition; S_n consists of $n! = 1 \cdot 2 \cdot \dots \cdot n$ elements denoted by $[i_1, \dots, i_n]$, where $i_k := \beta(k)$ and β is the bijection defining the given permutation.

Geometrically, the permutation group S_3 can be interpreted as the isometry group $\mathrm{Sym}(\triangle)$ of the equilateral triangle, while S_4 is isomorphic to the isometry group of the regular tetrahedron (as we shall see in the next chapter).

2.6.3. Universality theorem. It turns out that permutation groups and free groups have important "universality" properties.

Theorem 2.6.4. (i) *For any finite group G there exists a monomorphism of G into S_n for some n.*

(ii) *For any group G with a finite number n of generators there exists an epimorphism of the free group $\mathbb{F}_n$ onto G.*

Proof. (i) Let $|G| = n$ and $g_0 \in G$; then the mapping

$$\beta_{g_0} : G \to S_n \quad \text{given by} \quad G \ni g \mapsto gg_0 \in G$$

is a monomorphism. Indeed, it is obviously a homomorphism (indeed, we have $\beta_{g_0}\beta_{g_1} = \beta_{g_0 g_1}$, because both maps are given by the rule $g \mapsto gg_0g_1$). The homomorphism β_{g_0} is injective, because $g_0 g = g_0 g'$ implies $g = g'$ by the cancellation rule.

(ii) Let $g_1, \dots, g_n$ be a set of generators of G. Then the mapping

$$\alpha : \mathbb{F}[a_1, \dots, a_n] \to G \quad \text{given by} \quad \alpha(a_i) = g_i,\ i = 1, \dots, n,$$

is obviously a homomorphism. It is also surjective, because to each element $g_{i_1}^{\varepsilon_1} g_{i_2}^{\varepsilon_2} \dots g_{i_m}^{\varepsilon_m} \in G$, where the ε_i's are equal to ± 1, the mapping α takes the element $a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \dots a_{i_m}^{\varepsilon_m} \in \mathbb{F}[a_1, \dots, a_n]$. □

2.7. Group presentations

A presentation of a group is a way of defining the group by means of equations (called *defining relations*) in the generators of the group. This reduces concrete calculations in the group to the formal editing of words according to simple rules. The formal definition of the notion

of group presentation is easy to state but perhaps difficult to grasp, so we begin with some examples.

2.7.1. Examples of group presentations. (i) Consider all words in the three-letter alphabet $\{e, a, a^{-1}\}$, i.e., expressions such as $eaa^{-1}a$, $a^{-1}aeaaa$, etc. Let us say that two words are *equivalent* if one can transform one word into another by means of the *trivial relations* $aa^{-1} = e = a^{-1}a$ and $ae = ea = a$ and the relation $a^5 = 1$ (as usual, a^5 stands for $aaaaa$). This is obviously an equivalence relation in the technical sense, i.e., it is reflexive, symmetric, and transitive, so that the set of all words splits into equivalence classes. Define the product of two equivalence classes as the class containing the concatenation of any two elements of the given classes. It is easy to see that this product is well defined, i.e., does not depend on the choice of representatives in the classes. Obviously, there will be 5 equivalence classes (determined by the elements $a, a^2, a^3, a^4, a^5 = e$) and they form a group under the product operation defined above. The group obtained is clearly isomorphic to $\mathbb{Z}_5$.

(ii) Now consider words in the five-letter alphabet $\{e, s_1^{\pm}, s_2^{\pm 1}\}$. Let us say that two words are *equivalent* if one can be transformed into the other by means of the *trivial relations* (which we won't write out again) and the relations $s_1^2 = s_2^2 = e$ and $s_1 s_2 s_1 = s_2 s_1 s_2$ (the latter is known as the *Artin relation*) . Defining the product of the corresponding equivalence classes as in the previous example, we obtain a group which is isomorphic to S_3 (see Problem 2.9 below).

2.7.2. Formal definition. The definition of a group presentation is the following. An expression of the form

$$G = \langle g_1, \dots, g_n \ : R_1, \dots, R_k \rangle,$$

where $R_1, \dots, R_k$ are words in the alphabet

$$A = \{g_1, \dots, g_n g_1^{-1}, \dots, g_n^{-1}\},$$

is called a *presentation* of the group G; the words R_j are called *relators*; the group G is defined by its presentation as the quotient group

$$\mathbb{F}[g_1, \dots, g_n]/\{R_1, \dots, R_k\},$$

where $\{R_1, \dots, R_k\}$ is the minimal (by inclusion) normal subgroup of the free group $\mathbb{F}[g_1, \dots, g_n]$ containing the elements $R_1, \dots, R_k$.

This formal definition may be difficult to understand. But the notion of group presentation is simple. The elements of the group G that it defines are words in the alphabet A defined up to the trivial relations (see 2.6.1) and up to the *defining relations* $R_1 = e, \dots, R_k = e$; the product is concatenation (and is well defined).

Here are some examples:

(i) $\mathbb{Z}_m = \langle a : a^m \rangle$ is the m-element cyclic group;

(ii) $\mathbb{F}[g_1, \dots, g_n] = \langle g_1, \dots, g_n \ : \ \rangle$ is the free group on n generators (nothing appears after the colon in the angle brackets because the free group has no defining relations);

(iii) the permutation group on four elements can be presented as

$$S_4 = \langle s_1, s_2, s_3 : s_1^2, s_2^2, s_3^2,\ s_1 s_2 s_1^{-1} s_2^{-1},\ s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1},\ s_2 s_3 s_2 s_3^{-1} s_2^{-1} s_3^{-1} \rangle. \tag{2.1}$$

More details and examples appear in the problem section of this chapter.

2.8. Cayley's theorem

The following theorem (due to the British mathematician Arthur Cayley) shows that the notion of abstract group is not a real generalization: all groups are in fact transformation groups!

Theorem 2.8.1. *Any group G is a transformation group acting on the set G by right multiplication: $g \mapsto gg_0$ for any $g_0 \in G$.*

Proof. First, we must show that the assignment $g \mapsto gg_0$ is a bijection for any $g_0 \in G$. But this is obvious: it is injective (by the cancellation rule) and surjective (to any element $h \in G$ the element g_0 assigns the element hg_0^{-1}). Further, we must verify the transformation group axioms (see 1.3.1). This verification is also obvious: the transformations defined by elements of G are closed under composition (because so are elements of G) and under taking inverse elements

(the transformation inverse to the one given by g_0 is the one given by g_0^{-1}). □

Corollary 2.8.2. *Any group is a geometry in the sense of Klein* (*i.e., in the sense of formal definition given in* 1.4.1).

This corollary shows (as we mentioned previously) that the definition of geometry given in 1.4.1 is of course too general; additional restrictions on the set of elements and the transformation group are needed to obtain an object about which most mathematicians will agree that it is a *bona fide* geometry. However, there seems to be no formal agreement on this subject, so that the "additional restrictions" to be imposed are a matter of opinion, and we will not specify any (at least on the formal level) in this course.

2.9. Problems

2.1. Describe all the finite groups of order 6 or less and supply each with a geometric interpretation.

2.2. Describe all the (nontrivial) normal subgroups and the corresponding quotient groups of

(a) the isometry group of the equilateral triangle;

(b) the isometry group of the regular tetrahedron.

2.3. Let G be the motion group of the plane, P its subgroup of parallel translations, and R its subgroup of rotations with fixed center O. Prove that the subgroup P is normal and the quotient group G/P is isomorphic to R.

2.4. Prove that if the order of a subgroup is equal to half the order of the group (i.e., the subgroup is of *index* 2), then the subgroup is normal.

2.5. Find all the orbits and stabilizers of all the points of the group $G \subset S_{10}$ generated by the permutation

$$[5, 8, 3, 9, 4, 10, 6, 2, 1, 7] \in S_{10}$$

acting on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

2.6. Find the maximum order of elements in the group (a) S_5; (b) S_{13}.

2.7. Find the least natural number n such that the group S_{13} has no elements of order n.

2.8. Prove that the permutation group S_n is generated by the transposition

$$(1,2) := [2,1,3,4,\ldots,n]$$

and the cycle

$$(1,2,\ldots,n) := [2,3,\ldots,n,1].$$

2.9. Present the symmetry group of the equilateral triangle by generators and relations in two different ways.

2.10. How many homomorphisms of the free group in two generators into the permutation group S_3 are there? How many of them are epimorphisms?

2.11. Prove that the group presented as

$$\langle a, b \mid a^2 = b^n = a^{-1}bab = 1\rangle$$

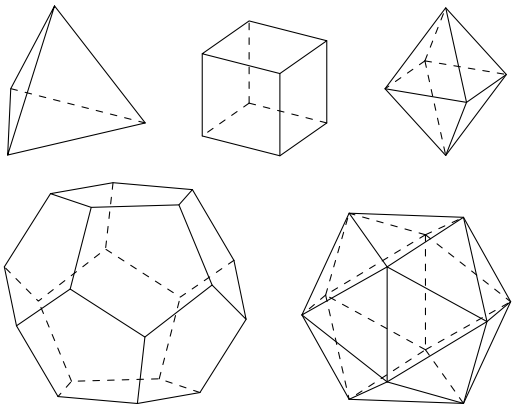
is isomorphic to the dihedral group $\mathbb{D}_n$ (defined in Chapter 3).

2.12. Show that if the elements a and b of a group satisfy the relations $a^5 = b^3 = 1$ and $b^{-1}ab = a^2$, then $a = 1$.

Chapter 3

Finite Subgroups of SO(3) and the Platonic Bodies

This chapter is devoted to the classification of regular polyhedra (the five "Platonic bodies") pictured below:



The proof of the classification theorem given here is based on group theory, more precisely on the study of finite subgroups of the isometry group of the two-dimensional sphere.

3.1. The Platonic bodies in art, philosophy, and science

The perfection of the shape of regular polyhedra attracted the great artist and thinker Leonardo da Vinci, who pictured them in various media. Figure 3.1 reproduces his engravings of two of them.

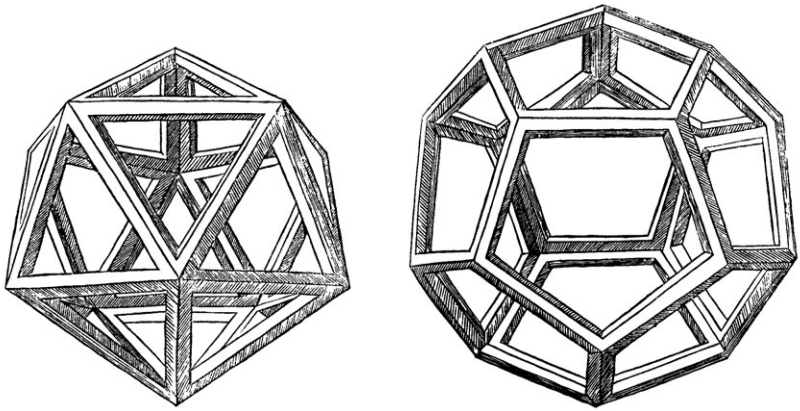


Figure 3.1. Da Vinci engravings: the icosahedron and dodecahedron.

Some philosophers and scientists felt an almost mystical attraction to these amazingly symmetric shapes. Thus the great astronomer Kepler believed that the distances from the planets to the Sun could be calculated from a system of nested inscribed Platonic bodies (see his weird engraving reproduced in Figure 3.2).

The engraving shows a cube inscribed in a sphere, then a smaller sphere inscribed in the cube, a tetrahedron inscribed in that second sphere, a third sphere inscribed in the tetrahedron, followed by successively inscribed sphere, dodecahedron, sphere, octahedron, sphere, icosahedron. Kepler claimed that the distances from the five planets to the Sun were proportional to the distances from the vertices of the five nested polyhedra to their common center of symmetry. He regarded this "discovery" as his main scientific achievement, far more important than the three fundamental astronomical laws that bear his name. Fortunately for his self-esteem, he did not live to see the

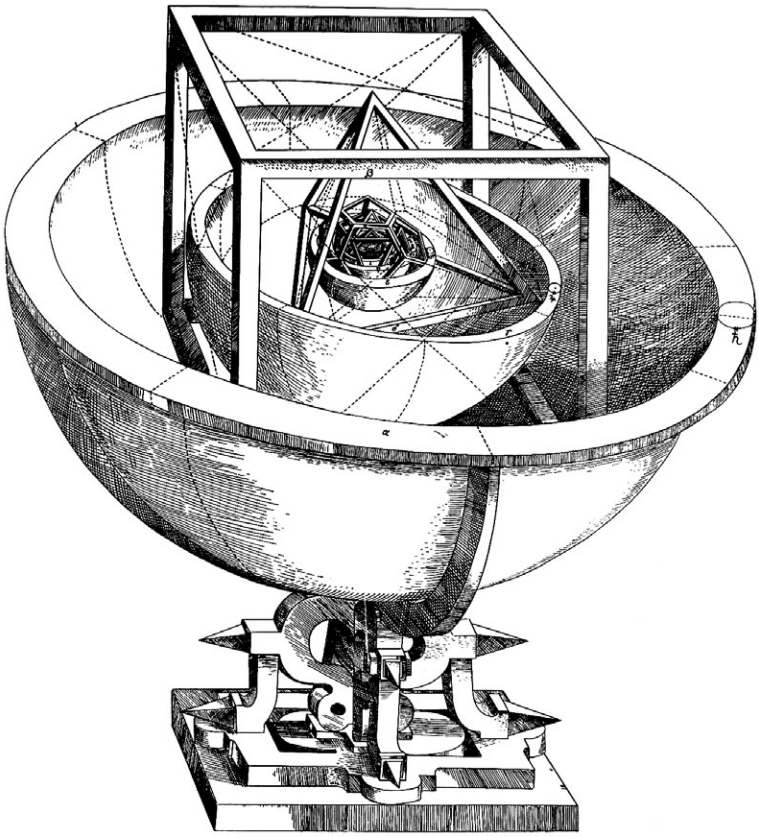


Figure 3.2. Kepler's theory of planetary orbits.

day when more exact measurements of the distances between the Sun and the planets showed that Kepler's theory was erroneous.

The five regular polyhedra were known to the ancient Greeks, in particular, to the philosopher Plato, who expressed his admiration for their unique perfection so beautifully that today they are often called "Platonic bodies". Of course Plato cannot be credited with their discovery (they were known before his time), but who the actual discoverers were is not clear. It is also unclear whether the

ancient Greeks had a proof of the fact that there are no other regular polyhedra, or indeed felt that such a proof was necessary. We can only conjecture that Archimedes had such a proof, or that it was possibly known to the Pythagorean school.

We do know whether Pythagoras was interested in the regular polyhedra in connection with his theory of the "singing spheres". In the 20th century, his theory was revived in the work of the German physicist Heisenberg, but the relevant ideas lie outside the scope of a mathematical textbook.

3.2. Finite subgroups of SO(3)

As we mentioned above, the main goal of this chapter is to prove that the only regular three-dimensional polyhedra are the five Platonic bodies. The proof that we give here is essentially group-theoretic (we reduce the classification problem of regular polyhedra to classifying finite subgroups of the special orthogonal group SO(3), or, which is the same thing, the group of motions of the sphere $\mathbb{S}^2$). This proof is quite natural and more geometric, in a deeper sense, than the tedious and eclectic space geometry proof anterior to the appearance of the notion of transformation group in mathematics.

Let us return to the geometry (briefly studied in Chapter 1, see 1.2.5) of the two-dimensional sphere

$$X = \mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$$

defined by the action of its isometry group $\mathrm{Sym}\,(\mathbb{S}^2)$. (In linear algebra courses this group is defined in a different (but equivalent) way, it is called the *orthogonal group*, and usually denoted by O(3).) Here we will be dealing with the subgroup of $\mathrm{O}(3) = \mathrm{Sym}\,(\mathbb{S}^2)$ consisting of rotations, namely the group $\mathrm{Rot}(\mathbb{S}^2)$ each element of which is a rotation of the sphere about some axis passing through the origin by some angle ϕ, $0 \leq \phi < 2\pi$. In linear algebra courses this group is defined in a different (but equivalent) way, is called the *special orthogonal group*, and is usually denoted by SO(3).

Our goal is to find the finite subgroups of SO(3). We begin with some examples of finite subgroups of O(3) and SO(3).

3.2.1. The monohedral group $\mathbb{Z}_n$ for any $n \geq 2$. Its n elements are rotations about an axis by angles of $2k\pi/n$, where $k = 0, \ldots, n-1$.

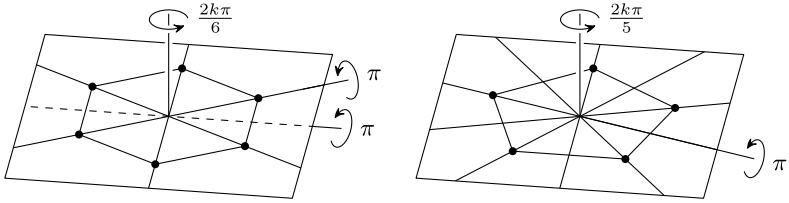


Figure 3.3. The dihedral group $\mathbb{D}_n$ for $n = 6$ and $n = 5$.

3.2.2. The dihedral group $\mathbb{D}_n$ for any $n \geq 2$. This $2n$-element group is the isometry group of the regular n-gon (lying in the horizontal plane Oxy and inscribed in the sphere $\mathbb{S}^2$); $\mathbb{D}_n$ consists of n rotations (by angles of $2k\pi/n$, $k = 0, 1, \ldots, n-1$) and n reflections in the horizontal lines passing through the center of the sphere, the vertices, and the midpoints of the sides (be careful: these lines are different when n is even or odd – look at the figure!). Note that the reflections of $\mathbb{D}_n$ in the horizontal lines are actually *rotations* by 180° in space about these lines.

3.2.3. The isometry group of the regular tetrahedron. It consists of 24 elements, it is denoted by $\mathrm{Sym}(\Delta^3)$, and its (12 element) rotation subgroup is:

$$\mathrm{Rot}(\Delta^3) = \mathrm{Sym}^+(\Delta^3) \subset \mathrm{Sym}(\Delta^3);$$

$\mathrm{Sym}(\Delta^3)$ consists of 8 rotations about 4 axes (containing one vertex) by angles of $2\pi/3$ and $4\pi/3$, of three rotations by π about axes joining the midpoints of opposite edges and of the identity. It is easy to see that $\mathrm{Sym}(\Delta^3)$ is isomorphic to the permutation group S_4. But here we think of this group geometrically, regarding the tetrahedron as inscribed in the sphere $\mathbb{S}^2$ and the elements of $\mathrm{Sym}(\Delta^3$ as acting on the sphere as well.

3.2.4. The isometry group $\mathrm{Sym}(I^3)$ of the cube. It has 48 elements (see 1.2.3); its rotation subgroup consists of 24 elements:

$$\mathrm{Rot}(I^3) = \mathrm{Sym}^+(I^3) \subset \mathrm{Sym}(I^3).$$

If we join the center of each of the 6 faces of the cube by segments to the four neighboring centers, we obtain the carcass of the *octahedron* dual to the cube (see Figure 3.4). The octahedron has 6 vertices and 8 triangular faces; its isometry group is obviously the same as that of the cube.

3.2.5. The isometry group $\mathrm{Sym}(\mathrm{Dod})$ of the dodecahedron. It has 120 elements and possesses a (60 element) rotation subgroup:

$$\mathrm{Rot}(\mathrm{Dod}) = \mathrm{Sym}^+(\mathrm{Dod}) \subset \mathrm{Sym}(\mathrm{Dod}).$$

The dodecahedron is the (regular) polyhedron (inscribed in the sphere $\mathbb{S}^2$) with 12 faces (congruent regular pentagons), 30 edges, and 20 vertices (see Figure 3.4). The existence of such a polyhedron will be proved at the end of this chapter. Joining the centers of the faces of the dodecahedron having a common edge (look at Figure 3.4 again), we get the *icosahedron* dual to the dodecahedron; it has 20 faces, 30 edges, and 12 vertices. Its transformation group is the same as that of the dodecahedron.

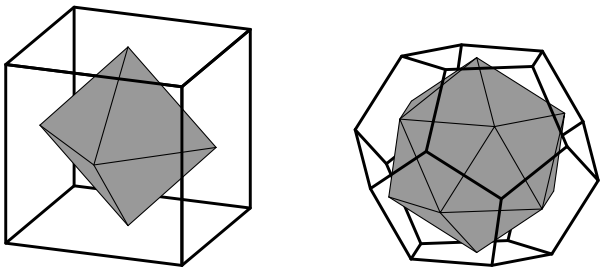


Figure 3.4. Dual pairs of regular polyhedra.

The following theorem states that SO(3) has no finite subgroups other than those listed above.

Theorem 3.2.6. *Any finite nontrivial subgroup G^+ of* $\mathrm{Sym}^+(\mathbb{S}^2) = \mathrm{SO}(3)$ *is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_n$, $n \geq 2$, (ii) $\mathbb{D}_n$, $n \geq 2$, (iii) $\mathrm{Rot}(\Delta^3)$, (iv) $\mathrm{Sym}^+(I^3)$, (v) $\mathrm{Sym}^+(\mathrm{Dod})$.

Proof. We know that any element of SO(3) (and hence of G^+) is a rotation about a diameter of the sphere $\mathbb{S}^2$ and has two fixed points (the ends of the diameter). Let F be the set of fixed points of the group G^+:

$$F = \big\{x \in \mathbb{S}^2 \mid \exists g \in G^+ \setminus \mathrm{id},\ xg = x\big\}.$$

For example, for the group $\mathbb{Z}_n$, F consists of two points, while for the rotation group $\mathrm{Rot}(\Delta^3)$ of the tetrahedron it has 14, namely the 4 vertices, the 4 intersection points of the axes of rotations of the faces with the sphere, and the 6 intersection points of the three axes of rotations passing through the midpoints of opposite edges of the tetrahedron.

Consider the (finite) geometry $(F : G^+)$ and let A be a set containing one point in each orbit of G^+ in F. First we claim that the number of points in F is

$$|F| = |A||G^+| - 2(|G^+| - 1).$$

The proof of this fact is the object of Problem 3.3 at the end of the present chapter. Using the class formula (1.2) from Chapter 1, we can write

$$|F| = \sum_{x \in A} \frac{|G^+|}{v(x)}, \quad \text{where} \quad v(x) := |\mathrm{St}(x)|.$$

Note that $v(x)$ is the order of the rotation subgroup of G^+ generated by the rotations about the axis containing x and its antipodal point. Replacing $|F|$ by its value found above and dividing by $|G^+|$, we obtain

$$\boxed{2 - \frac{2}{|G^+|} = \sum_{x \in A} \Big(1 - \frac{1}{v(x)}\Big)}, \tag{3.1}$$

or solving for $|G^+|$,

$$|G^+| = \Big[1 - \frac{1}{2} \cdot \sum_{x \in A} \Big(1 - \frac{1}{v(x)}\Big)\Big]^{-1}. \tag{3.2}$$

The left-hand side of the boxed formula is less than 2, but greater than or equal to 1; hence so is the sum in the right-hand side, and thus

the summation over A cannot contain 4 summands or more (because $v(x) \geq 2$); therefore there can be only 2 or 3 orbits of the action of G^+ on F.

First let us consider the case in which $|F| = 2$, i.e., when there is only one rotation axis (with intersection points x_1 and x_2 with the sphere). In that case there are two orbits in F, each consisting of one point, namely $\{x_1\}$ and $\{x_2\}$. Is such a situation possible? Of course it is, but only if G^+ consists of rotations about the unique axis x_1x_2. But then it follows that G^+ is isomorphic to $\mathbb{Z}_n$ for some $n \geq 2$. So the theorem is proved for the case $|F| = 2$. Note that in this case $v(x_1) = v(x_2) = n = |G^+|$.

It is easy to see that if the action of G^+ on F produces only two orbits, then the stabilizers of points from these two orbits have the same number of elements and we are in the case $|F| = 2$ considered above. Thus for the rest of the proof, we can assume that there are three orbits.

Denote by x_1, x_2, x_3 points of these three orbits, so that $A = \{x_1, x_2, x_3\}$, and denote by v_1, v_2, v_3 the values of $v(x)$ (the number of elements in the stabilizers, or which is the same thing, the order of the corresponding rotation axis) at these points, numbered so that $v_1 \leq v_2 \leq v_3$.

We can assume that $|F| > 2$ (the case $|F| = 2$ was considered above), i.e., there are two rotation axes or more; but then the composition of the two rotations gives a rotation about a third axis and so $|G^+| \geq 6$. We now claim that *there is an orbit with stabilizer equal to 2.*

Indeed, if, in contradiction with our claim, all the v_i were greater than 2, the right-hand side of formula (3.1) would be greater than or equal to 2, which we know is impossible.

Thus it remains to consider the situation in which $v_1 = 2$ and there are three orbits of the action of G^+ on F.

The rest of the proof is a case-by-case analysis of this situation depending on the possible values of the v_i. These values must satisfy relation (3.1), whose right-hand side is, as we remember, less than 2.

Thus we must have the inequality

$$3 - \frac{1}{2} - \frac{1}{v_2} - \frac{1}{v_3} < 2. \tag{3.3}$$

When is this inequality possible? Since v_2 and v_3 are both integers greater than or equal to 2, this can happen only in the cases 2–5 indicated in the following table (in it, the column for the number of elements of G^+ was filled by using formula (3.2)):

	v_1	v_2	v_3	$\lvert G^+\rvert$
case 1	n	n	-	n
case 2	2	2	n	$2n$
case 3	2	3	3	12
case 4	2	3	4	24
case 5	2	3	5	60

In the rest of the proof, we consider each case separately and distinguish (among the points of F) the vertices of a (possibly degenerate) polyhedron on which G^+ acts. We then show that this action is one of those listed in the claim of the theorem, i.e., the distinguished polyhedron either degenerates into a regular polygon or is the tetrahedron, or the cube, or the dodecahedron.

Case 1: This is the case in which $|F| = 2$; it was considered above, and we showed that it yields the group $\mathbb{Z}_n$, $n \geq 2$.

Case 2: Assume that $v_2 = 2$. Then we have two rotation axes l_1, l_2 of order 2, i.e., such that the rotation angle is 180°. Consider the line l_3 perpendicular to these two axes. One of its intersection points with the sphere will be x_3. Let n be the order of the axis l_3. It follows from formula (3.2) that the number of elements of G^+ is equal to $2n$. We can now specify the three orbits in F: the n-point orbit containing x_1, which lies in the plane perpendicular to l_3 passing through the center of the sphere, the n-point orbit containing x_2, lying in the same plane, and the 2-point orbit consisting of x_3 and its antipodal point. It is now clear that in our case G^+ is isomorphic to the dihedral group $\mathbb{D}_n$.

Case 3: $v_2 = v_3 = 3$. Then the number of elements of our group can be computed from formula (3.2), and is equal to 12. Consider

the axis of rotation l_3 passing through x_2; it is of order 3. Let x_3' and x_3'' be the two points to which the rotations about l_2 take the point x_3. The rotation about the axis l_1 containing the point x_1 is of order 2, hence at least one of the three points x_3, x_3', x_3'' must be taken to a point (which we denote by x_3''') that does not coincide with one of those three. Thus we obtain a tetrahedron x_3, x_3', x_3'', x_3''', which, as we will soon see, turns out to be regular. Taking the composition of the rotation about l_3 and the rotation about l_1, we get another rotation of order 3, from which we conclude that another face of the tetrahedron is an equilateral triangle, and therefore the tetrahedron x_3, x_3', x_3'', x_3''' is regular. Taking the composition of two order three rotations, we obtain another order two rotation and, continuing in the same vein, we describe all 12 rotations of G^+ and can conclude that G^+ is isomorphic to $\mathrm{Rot}(\Delta^3)$.

Case 4: Assume that $v_2 = 2$ and $v_3 = 4$. Here the strategy of proof is similar to the one in Case 3, except that now we find the 8 vertices of a cube (rather than those of a tetrahedron) among the points of F. To do this, we begin with the order four rotation, obtaining two squares inscribed in the sphere, then use the other rotations to show that the two squares are actually opposite faces of a cube, and finally verify that the 24 elements of G^+ are the symmetries of this cube, so that G^+ is isomorphic to $\mathrm{Sym}^+(I^3)$.

Case 5: Assume that $v_2 = 3$ and $v_3 = 5$. Here the strategy of proof is similar to that used in Cases 3 and 4, except that now we construct a dodecahedron from points of F and obtain an isomorphism between G^+ and $\mathrm{Sym}^+(\mathrm{Dod})$. We relegate the details to Problem 3.10.

Thus we see that the five cases correspond to the groups (i)–(v), respectively. The theorem is proved. □

3.2.7. Let us denote by $\widetilde{\mathbb{D}_n}$ the subgroup of SO(3) generated by the elements of $\mathbb{D}_n$ and the reflection ρ in the plane passing through the rotation axis of order n and one of the axes of order 2 in $\mathbb{D}_n$. Obviously the subgroup $\widetilde{\mathbb{D}_n}$ has $4n$ elements (because the compositions of ρ with different elements of $\mathbb{D}_n$ are all different from each other).

Note also that the subgroup of SO(3) generated by the elements of $\mathbb{Z}_n$ (interpreted as the motion group of the regular n-gon lying in the equatorial plane of the sphere and inscribed in it) and the reflection in a vertical plane passing through a vertex of the n-gon and the center of the sphere is $\mathbb{D}_n$.

Corollary 3.2.8. *Any finite subgroup G of* O(3) *is either isomorphic to one of the groups listed in Theorem* 3.2.6 *or to one of the following groups:*

(i) $\widetilde{\mathbb{D}}_n$, (ii) S_4, (iii) $\mathrm{Sym}(I^3)$, (iv) $\mathrm{Sym}(\mathrm{Dod})$.

Proof. Let G be a finite subgroup of SO(3) and let G^+ be its rotation subgroup. By Theorem 3.2.6, G^+ must be one of the five groups listed in the theorem. The whole group G is generated by the elements of G^+ and one reflection in a plane passing through the origin, so it must be one of the five groups listed in the statement of the corollary. □

3.3. The five regular polyhedra

A *regular polyhedron* is defined as a convex polyhedron inscribed in the sphere $\mathbb{S}^2$ such that

(i) all its faces are congruent regular polygons of k sides for some $k > 2$;

(ii) the endpoints of all the edges issuing from each vertex lie in one plane and form a regular l-gon for some $l > 2$.

Theorem 3.3.1. *There are exactly five different regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.*

Proof. This theorem follows from the corollary to Theorem 3.2.6. Indeed, the definition implies that the isometry group of a regular polyhedron is finite and therefore must be one of the groups listed in Theorem 3.2.6. The two "series" (i) and (ii) do not give any (non-degenerate) polyhedra (why?). In case (iii), we get the tetrahedron (because its symmetry group is isomorphic to the permutation group S_4). In case (iv), we get the cube and its dual, the octahedron, and in case (v), the dodecahedron and its dual, the icosahedron. □

Thus we obtain five geometries with three different group actions (tetrahedron, cube $\sim$ octahedron, dodecahedron $\sim$ icosahedron). To understand the group actions in these geometries, it is useful to have a look at their fundamental domains (Figure 3.5).

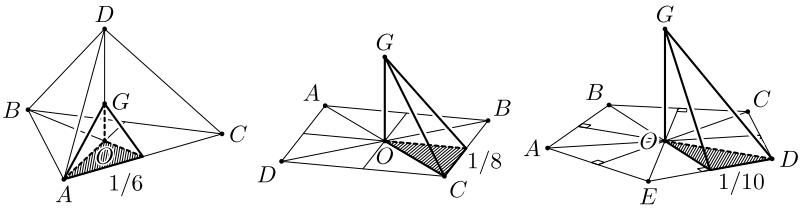


Figure 3.5. Fundamental domains of Platonic bodies.

In all five cases, their fundamental domains are pyramids with vertex the center of the body and base the fundamental domain (a right triangle in all five cases) of the isometry group of a face. These triangles have acute angles 30° (tetrahedron, octahedron, icosahedron), 45° (cube), 54° (dodecahedron).

3.4. The five Kepler cubes

Kepler observed that the cube can be inscribed in five different ways into the dodecahedron. Here we will perform the opposite construction: starting from the cube, we will construct a dodecahedron circumscribed to the cube. This will prove the existence of the dodecahedron.

Consider two copies $ABCDE$ and $A'B'C'D'E'$ of the regular pentagon with diagonals of length 1. Place these pentagons in the plane of the unit square $PQRS$ so that the diagonals BE and $B'E'$ are identified with PS and QR, respectively, and CD is parallel to $C'D'$. By rotating the pentagons in space about PS and QR, identify the sides CD and $C'D'$ above the square $PQRS$.

Now let $PQRS$ be the top face of the unit cube $PQRSP'Q'R'S'$. Place two more pentagons on the face $SRR'S'$ of the cube the same way as before, so that their parallel sides are parallel to SR. Now rotate these two pentagons until these parallel sides are identified.

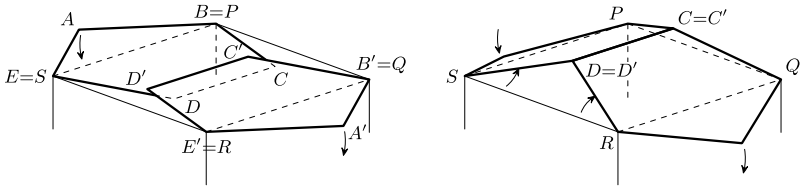


Figure 3.6. Constructing the dodecahedron.

Then it is not hard to prove that the upper endpoint of the identified segment will coincide with one of the endpoints of the common (identified) segment of the first two pentagons. Perform similar constructions on the other faces of the cube. The polyhedron thus obtained will be the dodecahedron.

3.5. Regular polyhedra in higher dimensions

In any dimensions $n > 3$, there is a classification theorem for regular n-dimensional polyhedra similar to that in three dimensions. Surprisingly, the number of types of polyhedra decreases with the increase of n, changing from five (in dim=3) and six (dim=4) to three (when dim$\geq$ 5). Thus, instead of the increased variety of regular bodies that might be expected in high dimensions, there are basically only three: the analogs of the tetrahedron, the cube, and the polyhedron dual to the cube.

In this section, after presenting the necessary definitions, we state the corresponding classification theorems without proof.

3.5.1. Examples and definitions. We begin with a simple example: the *four-dimensional cube*. In the Euclidean space $\mathbb{R}^4$, consider the 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$; their convex hull is by definition the 4-cube. A projection of the four-dimensional cube on the plane appears in Figure 3.7.

Even simpler (as its name indicates) is the regular *n-dimensional simplex* Δ^n, which is the n-dimensional analog of the tetrahedron, and is defined inductively: given the $(n-1)$-dimensional (regular) simplex Δ^{n-1} lying in $\mathbb{R}^{n-1}$, we construct a perpendicular from its center of gravity into the nth dimension (i.e., a line parallel to the

basis vector $(0,\dots,0,1)\in\mathbb{R}^n\supset\mathbb{R}^{n-1}$) and take for the $n+1$st vertex of our simplex the point whose distance from the n vertices of Δ^{n-1} is equal to the length of the edges of Δ^{n-1}. It is easy to see that the transformation group of Δ^n is the permutation group Σ_{n+1}.

Regular n-dimensional polyhedra are defined recursively. The recursion begins for $n=3$ and is that of a Platonic body (see Sec. 3.3 above). If regular $(n-1)$-dimensional polyhedra have been defined, we define a *regular n-dimensional polyhedron* as a convex polyhedron (inscribed in the sphere $\mathbb{S}^{n-1}:=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1^2+\dots+x_n^2=1\}$) such that

(i) all of its faces are congruent regular $(n-1)$-dimensional polyhedra;

(ii) the endpoints of all the edges issuing from each vertex lie in one hyperplane and form a regular $(n-1)$-dimensional polyhedron; all such polyhedra are congruent (but are not necessarily the same as those from item (i)).

To each regular polyhedron P, one can assign its *symbol*, defined (inductively) as the n-tuple of integers $(r_1,r_2,\dots,r_{n-1})$ in which r_1 is the number of edges of any one of the 2-dimensional faces Q of P, while $(r_2,\dots,r_{n-1})$ is the symbol of Q. For example, $(4,3,3)$ is the symbol of the four-dimensional cube, $(5,3)$ is that of the dodecahedron, $(3,3,3,3)$ that of the five-dimensional regular simplex.

One can define the *dual* to any regular polyhedron in the natural way (similarly to the way it is done in dimension 3). For example, the 5-simplex is dual to itself, while the dual to the 4-cube is the so-called *cocube*, which has the symbol $(3,3,4)$.

Theorem 3.5.2. *There are (up to homothety) six different regular polyhedra in dimension* 4*; their symbols are*

$$(3,3,3),(4,3,3),(3,3,4),(3,4,3),(5,3,3),(3,3,5).$$

The reader who wishes to find a proof of this theorem is referred to Problem 3.13, in which hints about the mysterious polyhedra with symbols $(3,4,3)$, $(5,3,3)$, $(3,3,5)$ appear, or to Figure 3.7, where their projections on the plane are shown.

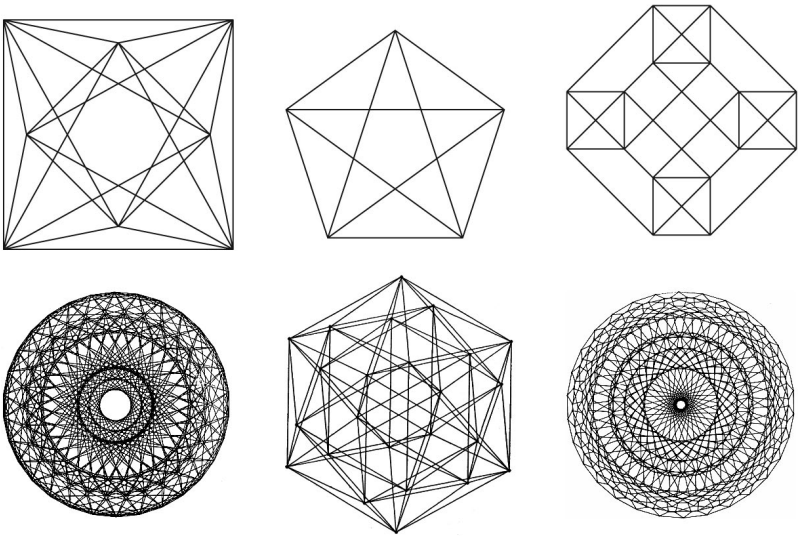


Figure 3.7. Regular 4-dimensional polyhedra.

Theorem 3.5.3. *In dimension $n \geq 5$ there are (up to homothety) three different regular polyhedra: the n-simplex, the n-cube, and the n-cocube; their symbols are*

$$(3, 3, \ldots, 3, 3), (4, 3, \ldots, 3, 3), (3, 3, \ldots, 3, 4).$$

We omit the proof (see [2]); the reader is also referred to Problem 3.14.

3.6. Problems

3.1. A regular pyramid of six lateral sides is inscribed in the sphere $\mathbb{S}^2$. Find its symmetry (i.e., isometry) group and its group of motions. How does your answer relate to the theorem on finite subgroups of SO(3)?

3.2. Answer the same questions as in Problem 3.1 for

(a) the regular prism of six lateral sides;

(b) the regular truncated pyramid of five lateral sides;

(c) the double regular pyramid of six lateral sides (i.e., the union of two regular pyramids of six lateral sides with common base and vertices at the poles of the sphere).

3.3. Let G^+ be a finite subgroup of SO(3) acting on the sphere $\mathbb{S}^2$ and F the set of all the points fixed by nontrivial elements of G^+; prove that F is invariant with respect to the action of G^+ and

$$|F| = |G^+| \cdot |A| - 2(|G^+| - 1),$$

where $A \subset F$ is a set containing exactly one point from each orbit of the action of G^+ on the set F.

3.4. Does the motion group of the cube have a subgroup isomorphic to the motion group of the regular tetrahedron?

3.5. Does the motion group of the dodecahedron have a subgroup isomorphic to the motion group of the cube?

3.6. In the motion group of the cube, find all groups isomorphic to $\mathbb{Z}_n$ and $\mathbb{D}_n$ for various values of n. Does it have any other subgroups?

3.7. Prove the existence of the dodecahedron in detail.

3.8. Given a cube inscribed in the sphere, let the set F consist of all the vertices of the cube, all the intersection points of the lines joining the centers of its opposite faces, and of the lines joining the midpoints of opposite edges, and let G^+ be the motion group of the cube. Prove that G^+ acts on F, find all the orbits of this action and the stabilizers of all the points of F. Compare your findings with the proof of Theorem 3.2.6 in Case 4.

3.9. Given a regular tetrahedron inscribed in the sphere, let the set F consist of all its vertices and of the lines joining the midpoints of the edges, and let G^+ be the motion group of the tetrahedron. Prove that G^+ acts on F, find all the orbits of this action and the stabilizers of all the points of F. Compare your findings with the proof of Theorem 3.2.6 in Case 3.

3.10. Given a dodecahedron inscribed in the sphere, let the set F consist of all the vertices of the dodecahedron, all the intersection points of the lines joining the centers of its opposite faces and of the

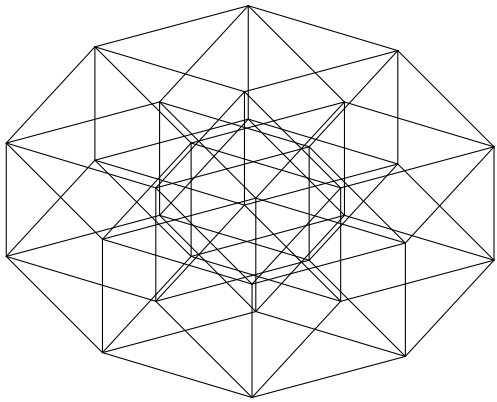


Figure 3.8. Projection of the edges of the 5-dimensional cube.

lines joining the midpoints of the edges, and let G^+ be the motion group of the dodecahedron. Prove that G^+ acts on F and complete the proof of Theorem 3.2.6 in Case 5.

3.11. Prove Theorem 3.2.6 in Case 4 by constructing an octahedron (instead of a cube) from the points of F.

3.12. Use your computer to produce a picture of the projection on an appropriately chosen two-dimensional plane of the five-dimensional cube. Compare with Figure 3.8.

3.13*. Prove the classification theorem for regular polyhedra in dimension four.

3.14*. Prove the classification theorem for regular polyhedra in dimension five.

Chapter 4

Discrete Subgroups of the Isometry Group of the Plane and Tilings

This chapter, just as the previous one, deals with a classification of objects, the original interest in which was perhaps more aesthetic than scientific, and goes back many centuries ago. The objects in question are regular tilings (also called tessellations), i.e., configurations of identical figures that fill up the plane in a regular way. Each regular tiling defines a geometry in the sense of Klein; it turns out that, up to isomorphism, there are 17 such geometries; their classification will be obtained by studying the corresponding transformation groups, which are discrete subgroups (see the definition in Section 4.3) of the isometry group of the Euclidean plane.

4.1. Tilings in architecture, art, and science

In architecture, regular tilings appear, in particular, as decorative mosaics (Figure 4.1) in the famous Alhambra palace (14th century Spain). According to M. Berger [2] and B. Grünbaum [7], part or all the 17 geometries mentioned above are realized by Alhambra mosaics. The reader can easily find beautiful color reproductions in the web by Googling "Alhambra mosaics".

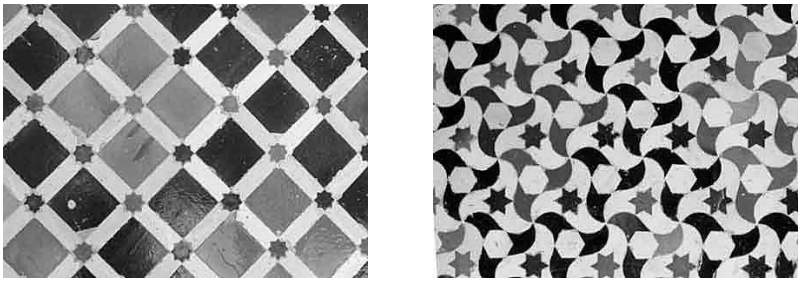


Figure 4.1. Two Alhambra mosaics.

In art, the famous Dutch artist M.C. Escher, known for his "impossible" paintings, used regular tilings as the geometric basis of his wonderful "periodic" watercolors, which can be easily found in the Internet, but not copied and reproduced. (Apparently, the Escher Foundation is more interested in making money than in popularizing the artist's work.) So we can show only the regular tilings underlying two of Escher's periodic watercolors (Figure 4.2).

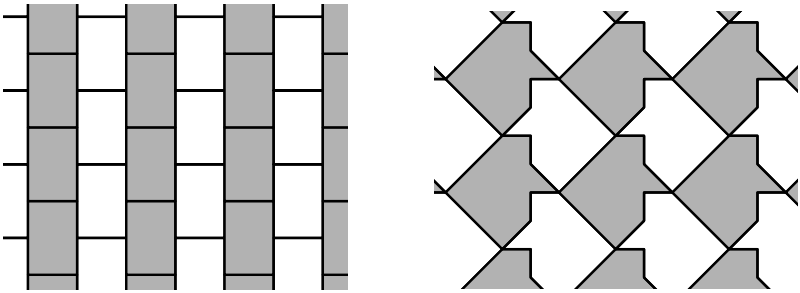


Figure 4.2. Two periodic tilings.

From the scientific viewpoint, not only regular tilings are important: it is possible to tile the plane by copies of one tile (or two) in an irregular (nonperiodic) way. It is easy to fill the plane with rectangular tiles of size say 10cm by 20cm in many nonperiodic ways. But the fact that $\mathbb{R}^2$ can be filled irregularly by nonconvex 9-gons is not obvious. Such an amazing construction, due to Vorderberg (1936), is shown in Figure 4.3. The figure indicates how to fill the plane by

copies of two tiles (their enlarged copies are shown separately; they are actually mirror images of each other) by fitting them together to form two spiraling curved strips covering the whole plane.

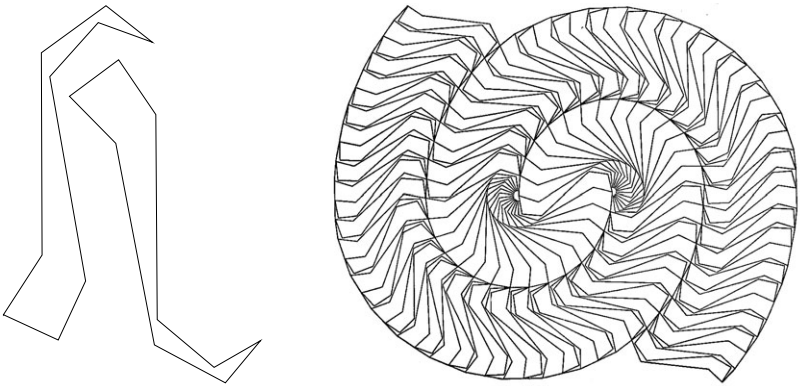


Figure 4.3. The Vorderberg tiling.

Somewhat later, in the 1960s, interest in irregular tilings was revived by the nonperiodic tilings due to the British mathematical physicist Roger Penrose, which are related to statistical models and the study of quasi-crystals. A version of a very famous Penrose tiling is shown in Figure 4.4 on the next page. The reader can find Penrose's original version by searching the web for "Penrose tiling" or (for a more artistic and amusing version), "Penrose chickens".

4.2. Tilings and crystallography

The first proof of the classification theorem of regular tilings (defined below, see 4.5.1) was obtained by the Russian crystallographer Fedorov in 1891. Mathematically, they are given by special discrete subgroups, called the *Fedorov groups*, of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane. As we mentioned above, there are 17 of them (up to isomorphism). The Fedorov groups act on the Euclidean plane, forming 17 different (i.e., nonisomorphic) geometries in the sense of Klein, which we call *tiling geometries*.

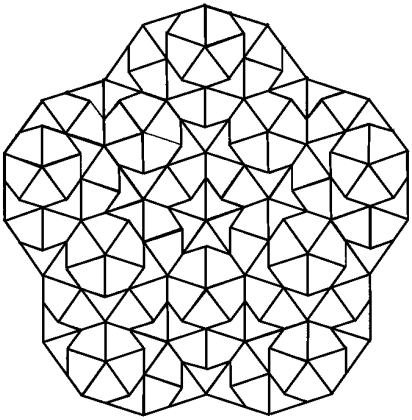


Figure 4.4. A Penrose tiling.

The proof given here, just as the one in the previous chapter, is group-theoretic, and is based on the study of discrete subgroups of the isometry group of the plane. In fact, the actual classification principle cannot be stated without using transformation groups, and at first glance it is difficult to understand how it came about that the architects of the Alhambra palace, five centuries before the notion of group appeared in mathematics, actually found most or all the 17 regular tilings (in this connection, see the article [7] by B. Grünbaum). Actually, this is not surprising: a deep understanding of symmetry suffices to obtain answers to an intuitively clear question, even if one is unable to state the question in the terminology of modern mathematics.

Less visual, but more important for the applications (crystallography) is the generalization of the notion of regular tiling to three dimensions: configurations of identical polyhedra filling $\mathbb{R}^3$ in a regular way. Mathematically, they are also defined by means of discrete subgroups called *crystallographic groups* of the isometry group of $\mathbb{R}^3$ and have been classified: there are 230 of them. Their study is beyond the scope of this book.

We are concerned here with the two-dimensional situation, and accordingly we begin by recalling some facts from elementary plane geometry, namely facts concerning the structure of isometries of the plane $\mathbb{R}^2$.

4.3. Isometries of the plane

Recall that by $\mathrm{Sym}(\mathbb{R}^2)$ we denote the group of isometries (i.e., distance-preserving transformations) of the plane $\mathbb{R}^2$, and by $\mathrm{Sym}^+(\mathbb{R}^2)$ its group of motions (i.e., isometries preserving orientation). Examples of the latter are parallel translations and rotations, while reflections in a line are examples of isometries which are not motions (they reverse orientation).

(We consider an isometry orientation-reversing if it transforms a clockwise oriented circle into a counterclockwise oriented one. This is not a mathematical definition, since it appeals to the physical notion of "clockwise rotation", but there is a simple and rigorous mathematical definition of orientation-reversing (-preserving) isometry in linear algebra courses, based on the sign ($\pm$) of the determinant of the corresponding linear map.)

Below we list some well-known facts about isometries of the plane; their proofs are relegated to exercises appearing at the end of the present chapter. These facts are also discussed in Section 0.9 of Chapter 0.

4.3.1. A classical theorem of elementary plane geometry says that any motion is either a *parallel translation* or a *rotation* (see Problem 4.1).

4.3.2. A less popular but equally important fact is that any orientation-reversing isometry is a *glide reflection*, i.e., the composition of a reflection in some line and a parallel translation by a vector collinear to that line (Problem 4.2).

4.3.3. The composition of two rotations is a rotation (except for the particular case in which the two angles of rotation are equal but opposite: then their composition is a parallel translation). In the general case, there is a simple construction that yields the center and

angle of rotation of the composition of two rotations (see Problem 4.3). This important fact plays the key role in the proof of the theorem on the classification of regular tilings.

4.3.4. The composition of a rotation and a parallel translation is a rotation by the same angle about a point obtained by shifting the center of the given rotation by the given translation vector (Problem 4.4).

4.3.5. The composition of two reflections in lines l_1 and l_2 is a rotation about the intersection point of the lines l_1 and l_2 by an angle equal to twice the angle from l_1 to l_2 (Problem 4.5).

4.4. Discrete groups and discrete geometries

The action of a group G on a space X is called *discrete* if none of its orbits possesses accumulation points, i.e., there are no points $x \in X$ such that any neighborhood of x contains infinitely many points belonging to one orbit. Here the word "space" can be understood as Euclidean space $\mathbb{R}^n$ (or as a subset of $\mathbb{R}^n$), but the definition remains valid for arbitrary metric and topological spaces.

A simple example of a discrete group acting on $\mathbb{R}^2$ is the group consisting of all translations of the form $k\,\vec{v}$, where v is a fixed nonzero vector and $k \in \mathbb{Z}$. The set of all rotations about the origin of $\mathbb{R}^2$ by angles which are integer multiples of $\sqrt{2}\,\pi$ is a group, but its action on $\mathbb{R}^2$ is not discrete (since $\sqrt{2}$ is irrational, orbits are dense subsets of circles centered at the origin).

4.5. The seventeen regular tilings

4.5.1. Formal definition. By definition, a *tiling* or *tessellation* of the plane $\mathbb{R}^2$ by a polygon T_0, the *tile*, is an infinite family $\{T_1, T_2, \dots\}$ of pairwise nonoverlapping (i.e., no two distinct tiles have common interior points) copies of T_0 filling the plane, i.e., $\mathbb{R}^2 = \bigcup_{i=1}^{\infty} T_i$.

For example, it is easy to tile the plane by any rectangle in different ways, e.g., as a rectangular lattice as well as in many irregular,

nonperiodic ways. Another familiar tiling of the plane is the *honeycomb lattice*, where the plane is filled with identical copies of the regular hexagon.

A polygon $T_0 \subset \mathbb{R}^2$, called the *fundamental tile*, determines a *regular tiling* of the plane $\mathbb{R}^2$ if there is a subgroup G (called the *tiling group*) of the isometry group $\mathrm{Sym}(\mathbb{R}^2)$ of the plane such that

(i) G acts discretely on $\mathbb{R}^2$, i.e., none of the orbits of G has accumulation points;

(ii) the images of T_0 under the action of G fill the plane, i.e.,

$$\bigcup_{g\in G} g(T_0) = \mathbb{R}^2;$$

(iii) the action of G is *transitive*, i.e., for $g, h \in G$ the images $g(T_0)$, $h(T_0)$ of the fundamental tile coincide if and only if $g = h$.

Actually, (ii) and (iii) imply (i), but we will not prove this (see the first volume of Berger's book [2], pp. 37–38 of the French edition).

The action of a tiling group $G \subset \mathrm{Sym}(\mathbb{R}^2)$ on the plane $\mathbb{R}^2$ is, of course, a geometry in the sense of Klein that we call the *tiling geometry* (or *Fedorov geometry*) of the group G.

4.5.2. Examples of regular tilings. Six examples of regular tilings are shown in Figure 4.5.

Given two tiles, there is one element of the transformation group that takes one to the other. The question marks show *how* the tiles are mapped to each other. (Without the question marks, the action of the transformation group would not be specified; see Problem 4.16.)

The first five tilings (a–e) are *positive*, i.e., they correspond to subgroups of the group $\mathrm{Sym}^+(\mathbb{R}^2)$ of motions (generated by all rotations and translations) of the plane (one-sided tiles slide along the plane). The sixth tiling (f) allows "turning over" the (two-sided) tiles.

Let us look at the corresponding tiling groups in more detail.

Theorem 4.5.3 (Fedorov, 1891)**.** *Up to isomorphism, there are exactly five different one-sided tiling geometries of the plane* $\mathbb{R}^2$*. They are shown in Figure* 4.5(a)–(e).

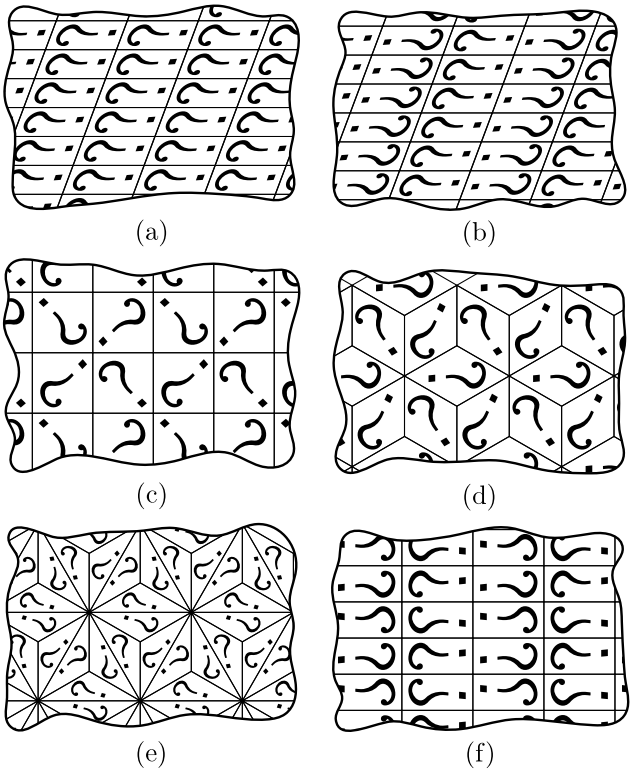


Figure 4.5. Six regular tilings of the plane.

Proof. Let G be a group of positive tilings. Consider its subgroup G_T of all the parallel translations in G.

Lemma 4.5.4. *The subgroup G_T is generated by two noncollinear vectors v and u.*

Proof. Arguing by contradiction, suppose that G_T is trivial (there are no parallel translations except the identity). Let r, s be any two (nonidentical) rotations with different centers. Then $rsr^{-1}s^{-1}$ is a nonidentical translation (to prove this, draw a picture). A contradiction. □

Now suppose that all the elements of G_T are translations generated by (i.e., proportional to) one vector v. Then it is not difficult to obtain a contradiction with item (ii) of the definition of regular tilings. □

Now if G contains no rotations, i.e., $G = G_T$, then we get the tiling (a). Further, if G contains only rotations of order 2, then it is easy to see that we get the tiling (b).

Lemma 4.5.5. *If G contains a rotation of order $\alpha \geq 3$, then it contains two more rotations (of some orders β and γ) such that*

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.}$$

Sketch of the proof. Let A be the center of a rotation of order α. Let B and C be the nearest (from A) centers of rotation not obtainable from A by translations. Then the boxed formula follows from the fact that the sum of angles of triangle ABC is π. The detailed proof of this lemma is the topic of one of the exercises. □

Since the three rotations are of order greater or equal to 3, it follows from the boxed formula that only three cases are possible.

	$1/\alpha$	$1/\beta$	$1/\gamma$
case 1	1/3	1/3	1/3
case 2	1/2	1/4	1/4
case 3	1/2	1/3	1/6

Studying these cases one by one, it is easy to establish that they correspond to the tilings (d), (c), (e) of Figure 4.5, respectively.

This concludes the proof of Theorem 4.5.3. □

In the general case (all tilings, including those by two-sided tiles), there are exactly 17 nonequivalent tilings. This was also proved by Fedorov. The 12 two-sided ones are shown on the next page.

We will not prove the second part of the classification theorem for regular plane tilings (it consists in finding the remaining 12 regular

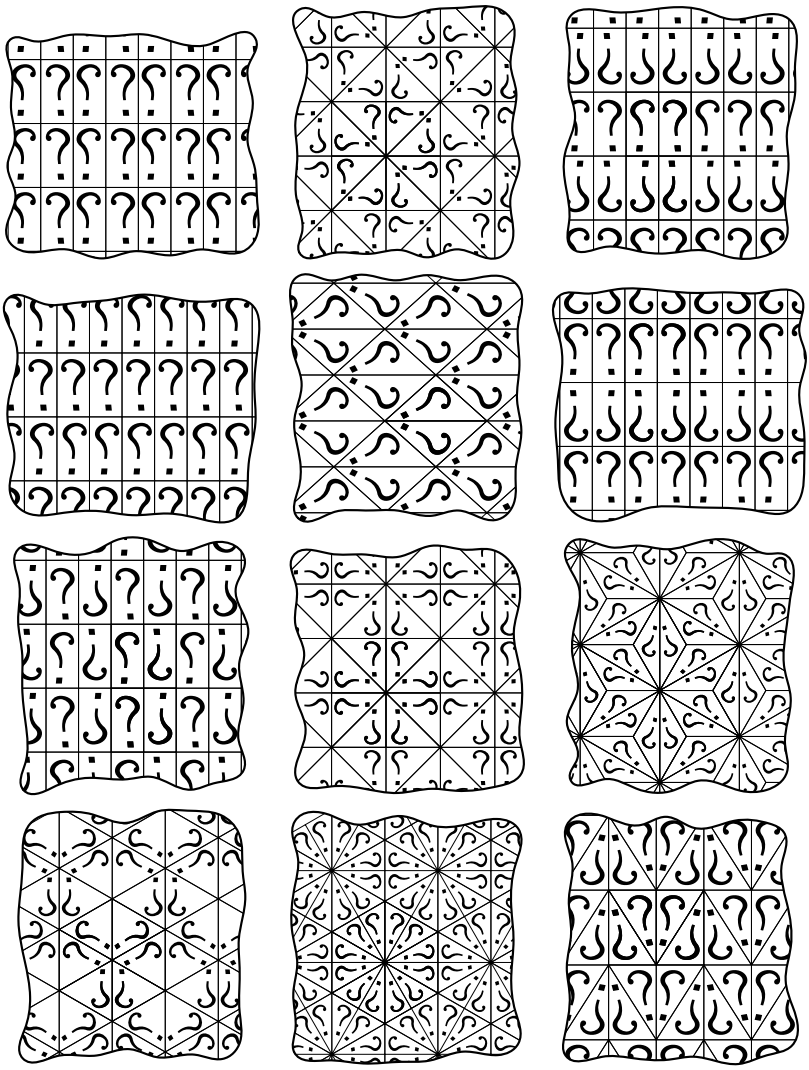


Figure 4.6. Two-sided regular tilings.

tilings, for which two-sided tiles are required). However, the reader can study examples of these 12 tilings by doing some of the exercises. Note that there is a nice website with beautiful examples of decorative patterns corresponding to the 17 regular tilings:

http://fac-web.spsu.edu/math/tile/symm/ident17.htm

One can also visit the Escher website.

4.6. The 230 crystallographic groups

The crystallographic groups are the analogs in $\mathbb{R}^3$ of the tiling groups in Euclidean space $\mathbb{R}^2$. The corresponding periodically repeated polyhedra are not only more beautiful than tilings, they are more important: the shapes of most of these polyhedra correspond to the shapes of real-life crystals. There are 230 crystallographic groups. The proof is very tedious: there are 230 cases to consider, in fact more, because many logically arising cases turn out to be geometrically impossible, and it lies, as we mentioned above, outside the scope of this book.

Those of you who would like to see some nontrivial examples of geometries corresponding to some of the crystallographic groups should look at Problem 4.5 and postpone their curiosity to the next chapter, where 4 examples of actual crystals will appear in the guise of Coxeter geometries. Another possibility is to consult the website http://webmineral.com/crystall.shtml or to google the words "crystallographic group".

4.7. Problems

4.1. Prove that any motion of the plane is either a translation by some vector v, $|v| \geq 0$, or a rotation r_A about some point A by a nonzero angle.

4.2. Prove that any orientation-reversing isometry of the plane is a glide reflection in some line L with glide vector u, $|u| \geq 0$, $u||L$.

4.3. Justify the following construction of the composition of two rotations $r = (a, \varphi)$ and (b, ψ). Join the points a and b, rotate the ray $[a, b\rangle$ around a by the angle $\varphi/2$, rotate the ray $[b, a\rangle$ around b by the

angle $-\psi/2$, and denote by c the intersection point of the two obtained rays; then c is the center of rotation of the composition rs and its angle of rotation is $2(\pi - \varphi/2 - \psi/2)$. Show that this construction fails in the particular case in which the two angles of rotation are equal but opposite, and then their composition is a parallel translation.

4.4. Prove that the composition of a rotation and a parallel translation is a rotation by the same angle and find its center of rotation.

4.5. Prove that the composition of two reflections in lines l_1 and l_2 is a rotation about the intersection point of the lines l_1 and l_2 by an angle equal to twice the angle from l_1 to l_2.

4.6. Indicate a finite system of generators for the transformation groups corresponding to each of the tilings shown in Figure 4.5 (a), (b), (c), (d), (e),(f).

4.7. Is it true that the transformation group of the tiling shown in Figure 4.5(b) is a subgroup of the one of Figure 4.5(c)?

4.8. Indicate the points that are the centers of the rotation subgroups of the transformation group of the tiling shown in Figure 4.5(c).

4.9. Write out a presentation of the isometry group of the plane preserving

(a) the regular triangular lattice;

(b) the square lattice;

(c) the hexagonal (i.e., honeycomb) lattice.

4.10. For which of the five Platonic bodies can a (countable) collection of copies of the body fill Euclidean 3-space (without overlaps)?

4.11. In the Internet, find the two Escher pictures schematically shown in Figure 4.2 and indicate to which of the 17 Fedorov groups they correspond.

4.12. Exactly one of the 17 Fedorov groups contains a glide reflection but no reflections. Which one?

4.13. Which two of the 17 Fedorov groups contain rotations by $\pi/6$?

4.14. Which three of the 17 Fedorov groups contain rotations by $\pi/2$?

4.15. Which five of the 17 Fedorov groups contain rotations by π only?

4.16. Rearrange the question marks in the tiling (c) of Figure 4.5 so as to make the corresponding geometry isomorphic to that of the tiling (a).

Chapter 5

Reflection Groups and Coxeter Geometries

In this chapter, as in the previous one, we study geometries defined by certain discrete subgroups of the isometry group of the plane (and, more generally, of n-dimensional space), namely the subgroups generated by reflections (called Coxeter groups after the 20th century Canadian mathematician who invented them). These geometries are perhaps not as beautiful as those studied in the previous two chapters, but are more important in the applications (in algebra and topology). On the other hand, they do have an aesthetic origin: what one sees in a kaleidoscope (a child's toy very popular before the computer era) is an instance of such a geometry. Following E.B. Vinberg, we call these geometries (in the two-dimensional case) kaleidoscopes. We prove the classification theorem for them in dimension 2 and state its generalization to higher dimensions without proof (using the notion of Coxeter scheme).

5.1. An example: the kaleidoscope

The kaleidoscope is a children's toy: bright little pieces of glass are placed inside a regular triangular prism and are multiply reflected by three mirrors forming the lateral faces of the prism.

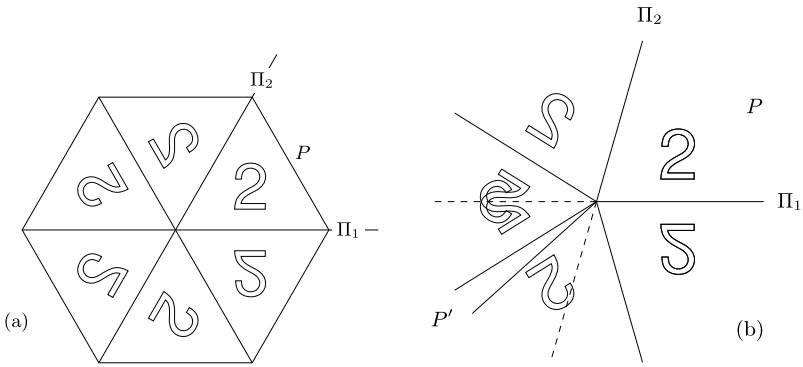


Figure 5.1. Geometry of the kaleidoscope.

Looking into the prism, you see a colorful repeated pattern: the picture in the triangle and its mirror images alternate, forming a hexagon (the union of six equilateral triangles), see Figure 5.1(a), surrounded by more equilateral triangles *ad infinitum*.

Mathematically, this is a two-dimensional phenomenon: the equilateral triangle forming the base of the prism is the fundamental domain of a discrete group acting on the plane of the base.

Now if the kaleidoscope is deformed (e.g., the angles between the faces of the prism are slightly changed), then the picture becomes fuzzy, no pattern can be seen. In such a situation, the images of the base triangle overlap infinitely many times (see Figure 5.1(b), the transformation group acting on the triangle is not discrete; we will not study this "bad" case: we only study the case of the "nice" kaleidoscope in dimension two and then generalize it to any dimension.

5.2. Coxeter polygons and polyhedra

Consider a dihedral angle $\alpha < \pi/2$ formed by two plane two-sided mirrors Π_1, Π_2. What will the observer O see? Any picture inside the angle will be reflected by Π_1; its image will in turn be reflected by the image of Π_1 by Π_2, and so on. At the same time, the picture inside the angle will be reflected by Π_2; its image will in turn be reflected by the image of Π_2 by Π_1, etc. Two cases are possible: either the

reflections coming from different sides will overlap (Figure 5.1(b)) or the reflected pictures will coincide (Figure 5.1(a)). Obviously, the pictures will coincide if (and only if) the angle α is of the form π/k, where $k = 2, 3, \ldots$.

Mathematically, this situation is the following. On the Euclidean plane, we take two straight lines forming the angle α and consider the group G of all transformations of the plane generated by the reflections in these two lines. Let F be the plane region bounded by the two rays forming the angle α. Obviously, no two regions $g(F)$ and $h(F)$, $g, h \in G$, $g \neq h$, overlap iff $\alpha = \pi/k$, where $k = 2, 3, \ldots$. In that case, G is the dihedral group $\mathbb{D}_k$.

Now suppose we are given a convex polygon F in the plane with vertex angles less than or equal to $\pi/2$. Consider the group G_F of transformations of the plane generated by reflections in the lines containing the sides of F. We say that G_F *acts transitively on* F if the images $g(F)$, $g \in G_F$, never overlap. A necessary condition for the transitive action of G_F on F is that all the vertex angles of F be of the form π/k for various values of k; this follows from the argument in the previous paragraph. Obviously, this condition is not sufficient.

The previous arguments are the motivation for the following definition. A convex polygon F is called a *Coxeter polygon* if all its vertex angles are of the form π/k for various values of $k = 2, 3, \ldots$ and it generates a transitive action of the group G_F. Coxeter polygons will be classified below – there are only four.

The above can be generalized to three-dimensional space. The corresponding definition is the following: a convex polyhedron is called a *Coxeter polyhedron* P if all its dihedral angles are of the form π/k for various values of $k = 2, 3, \ldots$ and it generates a transitive action of G_P, where G_P is the transformation group generated by the reflections in the planes containing the faces of P. Coxeter polyhedra will be classified below (there are seven).

5.3. Coxeter geometries on the plane

Let F be a Coxeter polygon in the plane $\mathbb{R}^2$. The *Coxeter geometry* with fundamental domain F is the geometry $(\mathbb{R}^2 : G_F)$, where G_F is

the group of transformations of the plane generated by the reflections in the lines containing the sides of the polygon F. The goal of this section is to classify all Coxeter geometries on the plane.

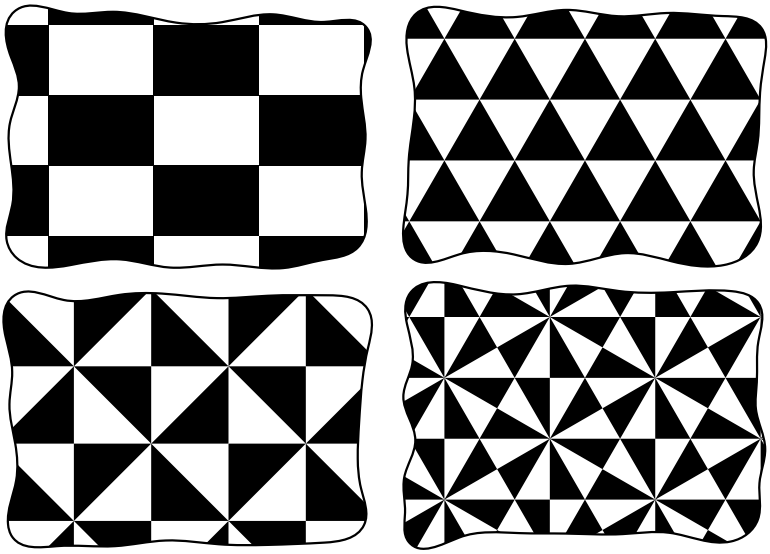


Figure 5.2. The four plane Coxeter geometries.

Theorem 5.3.1. *Up to isomorphism, there are four Coxeter geometries in the plane; their fundamental polygons are the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with angles* $\pi/3$ *and* $\pi/6$ (*see* Figure 5.2).

Proof. Let F be the fundamental polygon of a Coxeter geometry. If it has n sides, then the sum of its angles is $\pi(n-2)$ and so the average value of its angles is $\pi(1-2/n)$. Now n cannot be greater than 4, because F would then have an obtuse angle (and this contradicts the definition of Coxeter polygon). If $n = 4$, then all angles of F are $\pi(1-2/4) = \pi/2$ and F is a rectangle. Finally, if $n = 3$, and the angles of the fundamental triangle are $\pi/k, \pi/l, \pi/m$, then (since their sum is π) we obtain a Diophantine equation for k, l, m:

$$\boxed{\frac{1}{k}+\frac{1}{l}+\frac{1}{m}=1}.$$

This equation has three solutions: $(3,3,3)$, $(2,4,4)$, $(2,3,6)$. These solutions correspond to the three triangles listed in the theorem. □

5.4. Coxeter geometries in Euclidean space $\mathbb{R}^3$

5.4.1. In this section we study the Coxeter geometries in $\mathbb{R}^3$. A Coxeter polyhedron $F \subset \mathbb{R}^3$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^3$) with dihedral angles of the form π/k for various values of $k = 2, 3, \ldots$.

A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined just as in the case $d = 2$ (see Section 5.3).

Theorem 5.4.2. *There are seven Coxeter geometries in three-dimensional space; their fundamental polyhedra are the four right prisms over the rectangle, the equilateral triangle, the isosceles right triangle, and the right triangle with acute angles $\pi/3$ and $\pi/6$, and the three (nonregular) tetrahedra shown in* Figure 5.3.

It is not very difficult to prove that the seven polyhedra (listed in the theorem) indeed define Coxeter geometries. To prove that there are no other geometries, nontrivial information from linear algebra (in particular, the notion of Gramm matrix) is needed. Therefore, we omit the proof (see the book [4] or, for readers of Russian, a series of articles in *Matematicheskoe Prosveshchenie*, Ser. 3, no. 7, 2003).

A remark about terminology. The term "Coxeter geometry" is not a standard term. E.B. Vinberg uses the term "kaleidoscope" instead. Also, we do *not* use the term "Coxeter group" for the transformation group of a Coxeter geometry. This is because the expression "Coxeter group" is standardly used in the literature in a sense somewhat different from "transformation group of a Coxeter geometry".

Coxeter geometries are not only abstract mathematical objects, they are also important models in crystallography. For example, the polyhedron in Figure 5.3(b) is the crystal of ordinary salt, while the polyhedron in Figure 5.3(a) is a diamond crystal. Thus these polyhedra also appear as fundamental domains of the crystallographic groups mentioned in the previous chapter.

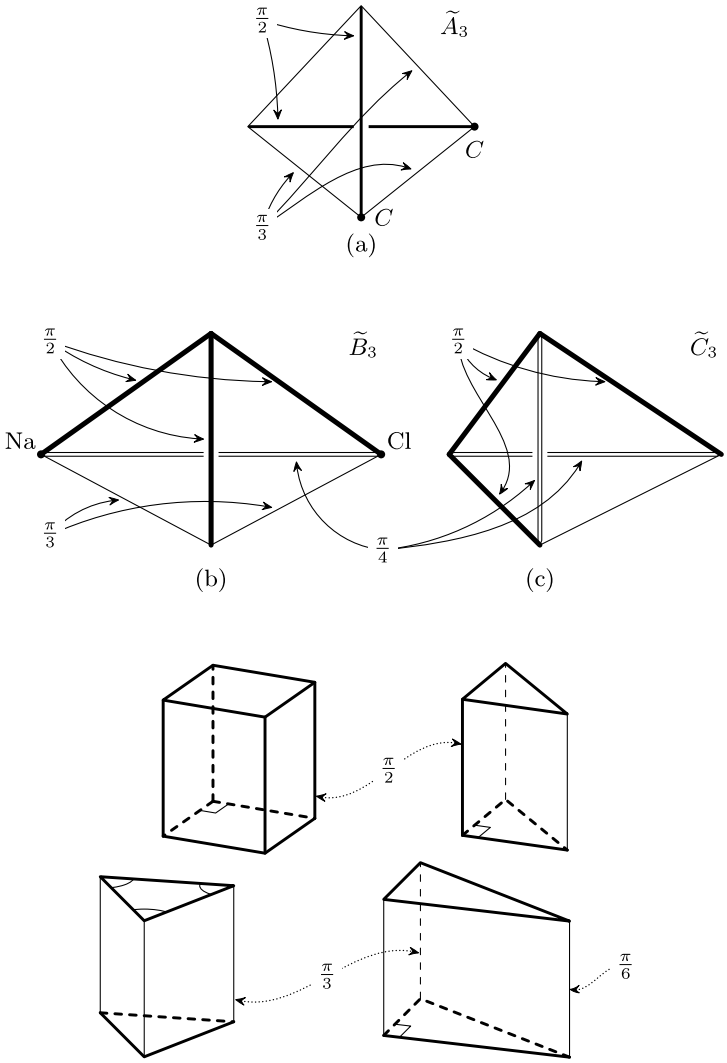


Figure 5.3. The seven Coxeter polyhedra in 3-space.

5.5. Coxeter schemes and the classification theorem

5.5.1. In this section we study the general case of a Coxeter geometry in $\mathbb{R}^d$ for an arbitrary positive integer d. A Coxeter polyhedron $F \subset \mathbb{R}^d$ is a convex polyhedron (i.e., the bounded intersection of a finite number of half-spaces in $\mathbb{R}^d$) with dihedral angles of the form π/k for various values of $k = 2, 3, \dots$ such that the reflections in the $(d-1)$-dimensionsal hyperplanes containing its faces generate a transitively acting group G_F. (The definition of the measure of a dihedral angle in Euclidean space of arbitrary dimension d appears in the linear algebra course.) A *Coxeter geometry* in $\mathbb{R}^d$ with fundamental polyhedron F is defined exactly as in the cases $d = 2$ and $d = 3$ (see Sections 5.3 and 5.4).

5.5.2. A *Coxeter scheme* is a graph (with integer weights on the edges) encoding a Coxeter polyhedron (in particular, polygons) in any dimension d. The scheme of a given Coxeter polyhedron is constructed as follows: its vertices correspond to the faces of the polyhedron, two vertices whose corresponding faces form an angle of π/m, $m \geq 3$, are joined by an edge with the weight $m-2$; if two faces are parallel, the corresponding vertices are joined by an edge with weight ∞. (Note that vertices corresponding to perpendicular edges are *not* joined by an edge.)

Graphically, instead of writing the weights $2, 3, 4$ on the edges of a scheme, we draw double, triple, quadruple edges; instead of writing ∞ on an edge, we draw a very thick edge.

For example, the Coxeter scheme of the rectangle consists of two components, each of which has two vertices joined by an edge with weight ∞, while the scheme of an equilateral triangle has three vertices joined cyclically by three edges with weights 1.

Theorem 5.5.3. *The Coxeter geometries in all dimensions are classified by the connected components of their Coxeter schemes listed in* Figure 5.4.

We omit the proof (see the book [4] or, for readers of Russian, the articles in the issue of *Matematicheskoe Prosveshchenie* cited above).

Many ideas of this (geometric) theory are similar to the very important (algebraic) theory of group representation (studied in advanced algebra courses and used in modern physics). In particular, the famous Dynkin diagrams are similar to Coxeter schemes and very similar notation (A_n, B_n, C_n, E_6, etc.) is used there.

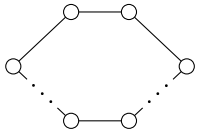
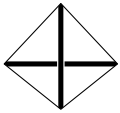
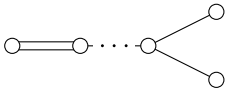
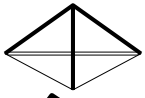
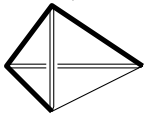
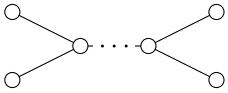
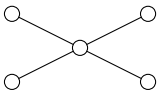
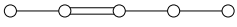
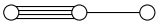
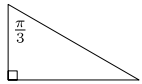
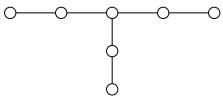
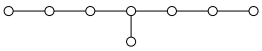
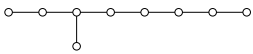
Name	Coxeter scheme	dim	#(faces)	view in $\mathbb{R}^3$
$\widetilde{A}_1$		1	2	
$\widetilde{A}_n$		$n-1$	n	
$\widetilde{B}_n$		$n-1$	n	
$\widetilde{C}_n$		$n-1$	n	
$\widetilde{D}_n$		$n-1$	$n \geq 5$	none!
$\widetilde{D}_4$		4	5	none!
$\widetilde{F}_4$		4	5	none!
$\widetilde{G}_2$		2	3	
$\widetilde{E}_6$				none!
$\widetilde{E}_7$				none!
$\widetilde{E}_8$				none!

Figure 5.4. Coxeter schemes for the Coxeter geometries.

5.6. Problems

5.1. Three planes P_1, P_2, P_3 passing through the z-axis of Euclidean space $\mathbb{R}^3$ are given. The angles between P_1 and P_2, P_2 and P_3 are α and β, respectively.

(a) Under what conditions on α and β will the group generated by reflections with respect to the three planes be finite?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

5.2. Three straight lines L_1, L_2, L_3 in the Euclidean plane form a triangle with interior angles α, β, and γ.

(a) Under what conditions on α, β, γ will the group generated by reflections with respect to the three lines be discrete?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

5.3. Consider the six lines $L_1, \ldots, L_6$ containing the six sides of a regular plane hexagon and denote by G the group generated by reflections with respect to these lines. Does this group determine a Coxeter geometry?

5.4. Let F be a Coxeter triangle, let s_1, s_2, s_3 be the reflections with respect to its sides, and G_F the corresponding transformation group.

(a) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F that leave a chosen vertex of F fixed.

(b) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F which are parallel translations.

Consider the three cases of different Coxeter triangles separately.

5.5. Draw the Coxeter schemes of

(a) all the Coxeter triangles;

(b) all the three-dimensional Coxeter polyhedra.

5.6. Prove that all the edges at each vertex of any three-dimensional Coxeter polyhedron lie on three straight lines passing through that vertex.

5.7. Let $(F : G_F)$ be a Coxeter geometry of arbitrary dimension. Prove that

(a) if $s \in G_F$ is the reflection in a hyperplane P, then, for any $g \in G_F$, gsg^{-1} is the reflection in the hyperplane gP;

(b) any reflection from the group G_F is conjugate to the reflection in one of the faces of the polyhedron F,

5.8. Describe some four-dimensional Coxeter polyhedron other than the four-dimensional cube.

5.9. (a) Does the transformation group generated by the reflections in the faces of regular tetrahedron define a Coxeter geometry?

(b) Same question for the cube.

(c) Same question for the octahedron.

(d) Same question for the dodecahedron.

Chapter 6

Spherical Geometry

So far we have studied finite and discrete geometries, i.e., geometries in which the main transformation group is either finite or discrete. In this chapter, we begin our study of infinite continuous geometries with spherical geometry, the geometry ($\mathbb{S}^2$:O(3)) of the isometry group of the two-dimensional sphere, which is in fact the subgroup of all isometries of $\mathbb{R}^3$ that map the origin to itself; O(3) is called the *orthogonal group* in linear algebra courses.

But first we list the classical continuous geometries that will be studied in this course. Some of them may be familiar to the reader, others will be new.

6.1. A list of classical continuous geometries

Here we merely list, for future reference, several very classical geometries whose transformation groups are "continuous" rather than finite or discrete. We will not make the intuitively clear notion of continuous transformation group precise (this would involve defining the so-called *topological groups* or even *Lie groups*), because we will not study this notion in the general case: it is not needed in this introductory course. The material of this section is not used in the rest of the present chapter, so the reader who wants to learn about

spherical geometry without delay can immediately go on to Section 6.3.

6.1.1. *Finite-dimensional vector spaces* over the field of real numbers are actually geometries in the sense of Klein (the main definition of Chapter 1). From that point of view, they can be written as

$$\boxed{(\mathbb{V}^n : \mathrm{GL}(n))},$$

where $\mathbb{V}^n$ denotes the n-dimensional vector space over $\mathbb{R}$ and $\mathrm{GL}(n)$ is the *general linear group*, i.e., the group of all nonsingular linear transformations of $\mathbb{V}^n$ to itself.

The subgeometry of $(\mathbb{V}^n : \mathrm{GL}(n))$ obtained by replacing the group $\mathrm{GL}(n)$ by its subgroup $\mathrm{O}(n)$ (consisting of orthogonal transformations) is called the *n-dimensional orthonormal vector space* and denoted

$$\boxed{(\mathbb{V}^n : \mathrm{O}(n))}.$$

These "geometries" are rather algebraic and are usually studied in linear algebra courses. We assume that the reader has some background in linear algebra and remembers the first basic definitions and facts of the theory.

6.1.2. *Affine spaces* are, informally speaking, finite-dimensional vector spaces "without a fixed origin". This means that their transformation groups $\mathrm{Aff}(n)$ contain, besides $\mathrm{GL}(n)$, all parallel translations of the space (i.e., transformations of the space obtained by adding a fixed vector to all its elements). We denote the corresponding geometry by

$$\boxed{(\mathbb{V}^n : \mathrm{Aff}(n))} \quad \text{or} \quad \boxed{(\mathbb{R}^n : \mathrm{Aff}(n))},$$

the latter notation indicating that the elements of the space are now regarded as *points*, i.e., the endpoints of the vectors (issuing from the origin) rather than the vectors themselves. This is a more geometric notion than that of vector space, but is also usually studied in linear algebra courses.

6.1.3. *Euclidean spaces* are geometries that we denote

$$\boxed{(\mathbb{R}^n : \mathrm{Sym}(\mathbb{R}^n))};$$

here $\mathrm{Sym}(\mathbb{R}^n)$ is the isometry group of Euclidean space $\mathbb{R}^n$, i.e., the group of distance-preserving transformations of $\mathbb{R}^n$. This group has, as a subgroup, the *orthogonal group* $\mathrm{O}(n)$ that consists of isometries leaving the origin fixed (the group $\mathrm{O}(n)$ should be familiar from the linear algebra course), but also contains the subgroup of parallel translations.

We assume that, for $n = 2, 3$, the reader knows Euclidean geometry from school (of course it was introduced differently, usually via some modification of Euclid's axioms) and is familiar with the structure of the isometry groups of Euclidean space for $n = 2, 3$.

The group $\mathrm{Sym}(\mathbb{R}^2)$ of isometries of the plane is generated by parallel translations, rotations, and mirror reflections (symmetries with respect to a line); it contains as a subgroup the *group of motions* of the plane (denoted by $\mathrm{Sym}^+(\mathbb{R}^2)$ and generated by rotations and translations).

The group $\mathrm{Sym}^+(\mathbb{R}^3)$ is generated by parallel translations, rotations about lines, and mirror reflections (symmetries with respect to planes); it contains as a subgroup the group of motions of 3-space (denoted by $\mathrm{Sym}^+(\mathbb{R}^3)$ and generated by rotations and translations). Elements of $\mathrm{Sym}(\mathbb{R}^3)$ not contained in $\mathrm{Sym}^+(\mathbb{R}^3)$, e.g., mirror reflections, are *orientation-reversing*, i.e., they transform a right hand into a left hand.

The reader who feels uncomfortable with elementary Euclidean plane and space geometry can consult Chapter 0. A rigorous axiomatic approach to Euclidean geometry in dimensions $d = 2, 3$ (based on Hilbert's axioms) appears in Appendix B.

Note that the transformation groups of these three geometries (vector spaces, affine and Euclidean spaces) act on the same space ($\mathbb{R}^n$ and $\mathbb{V}^n$ can be naturally identified), but the geometries that they determine are different, because the four groups $\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{Aff}(n)$, $\mathrm{Sym}(\mathbb{R}^3)$ are different. The corresponding geometries will not be studied in this course; traditionally, this is done in linear algebra

courses, and we have listed them here only to draw a complete picture of classical geometries.

Our list continues with three more classical geometries that we will study, at least in small dimensions (mostly in dimension 2).

6.1.4. *Hyperbolic spaces* $\mathbb{H}^n$ (called *Lobachevsky spaces* in Russia) are "spaces of constant negative curvature" (you will learn what this means much later, in differential geometry courses) with transformation group the isometry group of the hyperbolic space (i.e., the group of transformations preserving the "hyperbolic distance"). We will only study the hyperbolic space of dimension $n = 2$, i.e., the hyperbolic plane. Three models of $\mathbb{H}^2$ will be studied, in particular, the *Poincaré disk model*,

$$\boxed{(\mathbb{H}^2 : \mathcal{M})};$$

here $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is the open unit disk and $\mathcal{M}$ is the group of *Möbius transformations* (the definition appears in Chapter 7), which take the disk to itself.

We will also study two other models of hyperbolic plane geometry (the *half-plane model*, also due to Poincaré, and the *Cayley–Klein model*). A special chapter describes how attempts to prove Euclid's Fifth Postulate led to the appearance of hyperbolic plane geometry and the dramatic history of its creation by Gauss, Lobachevsky, and Bolyai.

6.1.5. *Elliptic spaces* $\mathbb{E}\mathrm{ll}^n$ are "spaces of constant positive curvature" (what this means is explained in differential geometry courses). We will only study the two-dimensional case, i.e., the elliptic plane, in the present chapter after we are done with *spherical geometry*, which is the main topic of this chapter, but can also be regarded as the principal building block of elliptic plane geometry.

6.1.6. *Projective spaces* $\mathbb{R}P^n$ are obtained from affine spaces by "adding points at infinity" in a certain way, and taking, for the transformation group, a group of linear transformations on the so-called "homogeneous coordinates" of points $(x_1 : \cdots : x_n : x_{n+1}) \in \mathbb{R}P^n$.

We can write this geometry as

$$\boxed{(\mathbb{R}P^n : \mathrm{Proj}(n))}.$$

For arbitrary n, projective geometry is usually studied in linear algebra courses. We will study the *projective plane* $\mathbb{R}P^2$ in this course, and only have a quick glance at projective space $\mathbb{R}P^3$ (see Chapter 12).

6.2. Some basic facts from Euclidean plane geometry

Here we list several fundamental facts of Euclidean plane geometry (including modern formulations of some of Euclid's postulates) in order to compare and contrast them with the corresponding facts of spherical, elliptic, and hyperbolic geometry.

I. *There exists a unique* (*straight*) *line passing through two given distinct points.*

II. *There exists a unique perpendicular to a given line passing through a given point.* (A *perpendicular* to a given line is a line forming four equal angles, called *right angles*, with the given one.)

III. *There exists a unique circle of given center and given radius.*

IV. *Given a point on a line and any positive number, there exist exactly two points on the line whose distance from the given point is equal to the given number.*

V. *There exists a unique parallel to a given line passing through a given point not on the given line.* (A *parallel* to a given line is a line without common points with the given one.) This is the modern version of Euclid's Fifth Postulate, sometimes described as the single most important and controversial scientific statement of all time.

VI. *The parameters of a triangle ABC, namely the angles α, β, γ at the vertices A, B, C and the sides a, b, c opposite to these vertices, satisfy the following formulas.*

(i) *Angle sum formula:* $\alpha + \beta + \gamma = \pi$.

(ii) *Sine formula:*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(iii) *Cosine formula:* $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

6.3. Lines, distances, angles, polars, and perpendiculars

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\};$$

our present aim is to study the geometry $(\mathbb{S}^2 : \mathrm{O}(3))$, where $\mathrm{O}(3)$ is the orthogonal group (i.e., the group of isometries of $\mathbb{R}^3$ leaving the origin in place), which obviously takes $\mathbb{S}^2$ to itself.

6.3.1. Basic definitions. By a *line* on the sphere we mean a great circle, i.e., the intersection of $\mathbb{S}^2$ with a plane passing through the sphere's center. For example, the equator of the sphere, as well as any meridian, is a line. The *angle* between two lines is defined as the dihedral angle (measured in radians) between the two planes containing the lines. For example, the angle between the equator and any meridian is $\pi/2$. The *distance* between two points A and B is defined as the measure (in radians) of the angle AOB. Thus the distance between the north and south Poles is π, the distance between the south pole and any point on the equator is $\pi/2$.

Obviously, *the transformation group* $\mathrm{O}(3)$ *preserves distances between points.* It can also be shown (we omit the proof) that, conversely, $\mathrm{O}(3)$ can be characterized as the group of distance-preserving transformations of the sphere (distance being understood in the spherical sense, i.e., as explained above).

6.3.2. Poles, polars, perpendiculars, circles. Let us look at the analogs in spherical geometry of the Euclidean postulates.

I$_S$**.** *There exist a unique line passing through two given distinct points, except when the two points are antipodal, in which case there are infinitely many.* All the meridians joining the two poles give an example of this exceptional situation.

II$_S$. *There exists a unique perpendicular to a given line passing through a given point, except when the point lies at the intersection of the perpendicular constructed from the center O of the sphere to the plane in which the line lies, in which case there are infinitely many such perpendiculars.* The exceptional situation is exemplified by the equator and, say, the north pole: all the meridians (which all pass through the pole) are perpendicular to the equator (Figure 6.1).

More generally, the *polar* of a point P is the (spherical) line obtained by cutting the sphere by the plane passing through O and perpendicular to the (Euclidean!) straight line PO. Conversely, given a (spherical) line l, the *poles* of that line are the two antipodal points P_l and P'_l for which the (Euclidean) line $P_lP'_l$ is perpendicular to the plane determined by l. The assertion II$_S$ may now be restated as follows: *there exists a unique perpendicular to a given line passing through a given point, except when the point is a pole of that line, in which case all the lines passing through the pole are perpendicular to the given line.*

III$_S$. *There exists a unique circle of given center C and given radius ρ, provided $0 < \rho < \pi$.* It is defined as the set of points whose (spherical) distance from C is equal to ρ. It is easy to see that any (spherical) circle is actually a Euclidean circle, namely the one obtained as the intersection of the sphere with the plane perpendicular to the Euclidean line OC and passing through the point I on that line

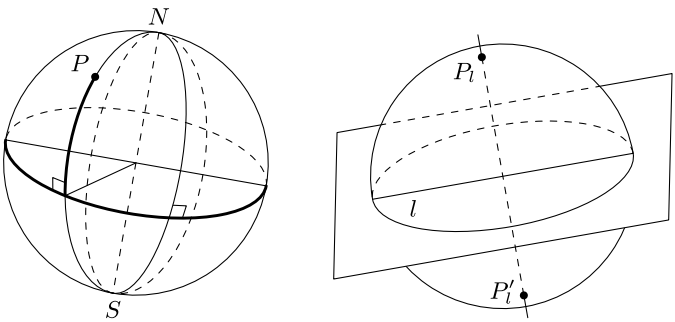


Figure 6.1. Perpendiculars, poles, and polars.

such that $OI = \cos\rho$. Note that the radius of the Euclidean circle will be less than ρ.

Given a spherical circle of center C and radius ρ, note that it can be regarded as the circle of radius $\pi - \rho$ and center C', where C' is the antipode of C. Further, note that the longest circle centered at C is the polar of the point C; its radius is $\pi/2$.

IV$_S$. Given a point on a line and any positive number, *there exist exactly two points on the line whose distance from the given point equals the given number, provided the number is less than* π.

V$_S$. Any two lines intersect in two *antipodal* points, i.e., in two points symmetric with respect to the center of the sphere $\mathbb{S}^2$. Therefore *there are no parallel lines in spherical geometry*. If two points A, B are not antipodal, then there is only one line joining them and one shortest line segment with endpoints at A and B. For opposite points, there is an infinity of lines joining them (for the north and south poles, these lines are the meridians).

6.3.3. Lines as shortest paths. It is proved in differential geometry courses that *spherical lines are geodesics*, i.e., they are the shortest paths between two points. To do this, one defines the length of a curve as a curvilinear integral and uses the calculus of variations to show that the curve (on the sphere) of minimum length joining two given points is indeed the arc of the great circle containing these points.

6.4. Biangles and triangles in $\mathbb{S}^2$

6.4.1. Biangles. Two lines l and m on the sphere intersect in two (antipodal) points P and P' and divide the sphere into four domains; each of them is called a *biangle*, it is bounded by two halves of the lines l and m, called its *sides*, and has two *vertices* (the points P and P'). The four domains form two congruent pairs; two biangles from a congruent pair touch each other at the common vertices P and P', and have the same angle at P and P'. The main parameter of a biangle is the measure α of the angle between the lines that determine it; if $\alpha \neq \pi/2$, the two biangles not congruent to the biangle of measure α are called *complementary*, their angle is $\pi - \alpha$. Note that the angle

measure α determines the corresponding biangle up to an isometry of the sphere.

6.4.2. Areas of figures on the sphere. In order to correctly measure areas of figures on the plane, on the sphere, or on other surfaces, one must define what an area is, specify what figures are measurable (i.e., possess an area), and devise methods for computing areas. For the Euclidean plane, there are several approaches to area: many readers have probably heard of the theory of *Jordan measure*; more advanced readers may have studied *Lebesgue measure*; readers who have taken multivariable calculus courses know that areas may be computed by means of *double integrals*.

In this book, we will not develop a rigorous measure theory for the geometries that we study. In this subsection, we merely sketch an axiomatic approach for determining areas of spherical figures; this approach is similar to Jordan measure theory in the Euclidean plane. The theory says that there is a family of sets in $\mathbb{S}^2$, called *measurable*, satisfying the following axioms.

(i) *Invariance.* Two congruent measurable figures have the same area.

(ii) *Normalization.* The whole sphere is measurable and its area is 4π.

(iii) *Countable additivity.* If a measurable figure F is the union of a countable family of measurable figures $\{F_i\}$ without common interior points, then its area is equal to the sum of areas of the figures F_i.

An obvious consequence of these axioms is that the area of the northern hemisphere is 2π, while each of the triangles obtained by dividing the hemisphere into four equal parts is of area $\pi/2$.

6.4.3. Area of the biangle. From the axioms formulated in the previous subsection, it is easy to deduce that the area $S_{\pi/2}$ of a biangle with angle measure $\pi/2$ is π. Indeed, the sphere is covered by four such nonoverlapping biangles, which are congruent to each other; they have the same area by (i), the sum of their areas is that of the sphere by (iii), and the latter is 4π by (ii), whence $S_{\pi/2} = (4\pi)/4 = \pi$.

For the case in which the angle measure α of a biangle is a rational multiple of π, a similar argument shows that

$$\boxed{S_\alpha = 2\alpha}. \tag{6.1}$$

This formula is actually true for any α, but for the case in which π/α is irrational, its proof requires a passage to the limit based on an additional "continuity axiom" that we have not explicitly stated. We therefore omit the proof, but will use the above formula for all values of α in what follows.

6.4.4. Area of the triangle. Let A, B, C be three distinct points of $\mathbb{S}^2$, no two of which are opposite. The union of the shortest line segments joining the points A and B, B and C, C and A is called the *triangle* ABC. For a triangle ABC, we always denote by α, β, γ the measure of the angles at A, B, C, respectively, and by a, b, c the lengths of the sides opposite to A, B, C (recall that the length a of BC is equal to the measure of the angle BOC in $\mathbb{R}^3$).

Theorem 6.4.5. *The area S_{ABC} of a spherical triangle with angles α, β, γ is equal to*

$$\boxed{S_{ABC} = \alpha + \beta + \gamma - \pi}.$$

Proof. There are 12 spherical biangles formed by pairs of lines AB, BC, CA. Choose six of them, namely those that contain triangle ABC or the antipodal triangle $A_1B_1C_1$ formed by the three points antipodal to A, B, C. Denote their areas by

$$S_I, S_I', S_{II}, S_{II}', S_{III}, S_{III}'.$$

Each point of the triangles ABC and $A_1B_1C_1$ is covered by exactly three of the chosen six biangles, while the other points of the sphere are covered by exactly one such biangle (we ignore the points on the lines). Therefore, using relation (6.1), we can write

$$\begin{aligned} 4\pi &= S_I + S_I' + S_{II} + S_{II}' + S_{III} + S_{III}' - 2S_{ABC} - 2S_{A_1B_1C_1} \\ &= 2\alpha + 2\beta + 2\gamma + 2\alpha + 2\beta + 2\gamma - 2S_{ABC} - 2S_{A_1B_1C_1} \\ &= 4(\alpha + \beta + \gamma) - 4S_{ABC}, \end{aligned}$$

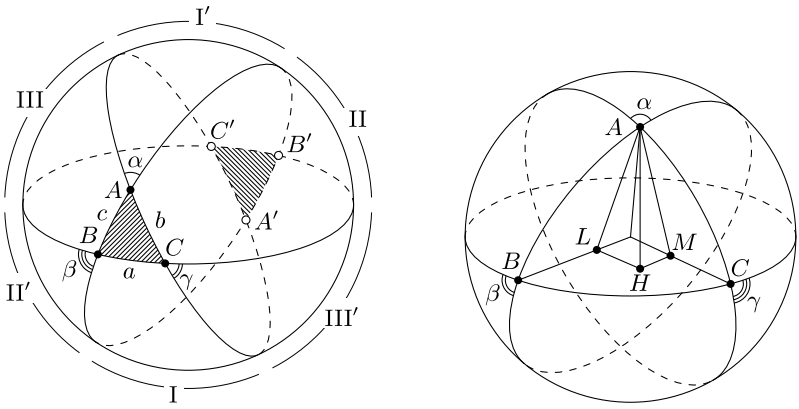


Figure 6.2. Area and sine theorem for the triangle.

because the two triangles ABC and $A_1B_1C_1$ have the same area (since they are congruent). Clearly, the previous formula implies the required equality. □

This theorem has the following fundamental consequence.

Corollary 6.4.6. *The sum of angles of any spherical triangle is more than π.*

The analog of the sine formula for the Euclidean triangle is the following statement about spherical triangles.

Theorem 6.4.7 (The spherical sine theorem)**.**

$$\boxed{\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}}.$$

In order to establish this formula, we will use the following statement, sometimes called the "theorem of the three perpendiculars".

Lemma 6.4.8. *Let $A \in \mathbb{R}^3$ be a point outside a plane $\mathcal{P}$, let K be its perpendicular projection on $\mathcal{P}$ and let L be its perpendicular projection on a line l contained in $\mathcal{P}$. Then KL is perpendicular to l.*

Proof of the lemma. The line l is perpendicular to the plane AKL because it is perpendicular to two nonparallel lines of AKL, namely

to AL and AK (to the latter since AK is orthogonal to any line in $\mathcal{P}$). Therefore l is perpendicular to any line of the plane AKL, and in particular, to LK. □

Proof of the theorem. Let H be the projection of A on the plane BOC, let L and M be the projections of A on the lines OB and OC. Then by the lemma, L and M coincide with the projections of H on OB and OC. Therefore,

$$AH = LA\sin\beta = \sin c\sin\beta, \quad AH = MA\sin\gamma = \sin b\sin\gamma.$$

Thus, $\sin b : \sin\beta = \sin c : \sin\gamma$. Similarily, by projecting C on the plane AOB and arguing as above, we obtain $\sin b : \sin\beta = \sin a : \sin\alpha$. This is the required equality. □

6.5. Other theorems about triangles

In this section, we state a few more theorems about spherical triangles. Their proofs are relegated to the problems appearing at the end of this chapter.

Theorem 6.5.1 (First cosine theorem)**.**

$$\boxed{\cos a = \cos b\cos c + \sin b\sin c\cos\alpha}.$$

Theorem 6.5.2 (Second cosine theorem)**.**

$$\boxed{\cos\alpha + \cos\beta\cos\gamma = \sin\beta\sin\gamma\cos\alpha}.$$

Corollary 6.5.3 (Analog of the Pythagorean theorem)**.** *If triangle ABC has a right angle at C, then*

$$\boxed{\cos c = \cos a\cos b}.$$

Theorem 6.5.4. *The medians of any triangle intersect at a single point.*

Theorem 6.5.5. *The altitudes of any triangle intersect at a single point.*

6.6. Coxeter triangles on the sphere $\mathbb{S}^2$

We will not develop the theory of tilings on the sphere $\mathbb{S}^2$ and Coxeter geometry on the sphere in full generality, but only consider *Coxeter triangles*, i.e., spherical triangles all of whose angles are of the form π/m, $m = 2, 3, \ldots$. It follows from Theorem 6.4.5 that any spherical Coxeter triangle $(\pi/p, \pi/q, \pi/r)$, N copies of which cover the sphere, must satisfy the Diophantine equation

$$N/p + N/q + N/r = N + 4.$$

The transformation group of the corresponding Coxeter geometry is finite, and so Theorem 3.2.6 tells us what group it has to be: it must be either one of the dihedral groups, or the tetrahedral, hexahedral, or dodecahedral group. The dihedral groups yield an obvious infinite series of tilings, one of which is shown in Figure 6.3.

The three other groups yield three possibilities for N, namely $N = 24, 48, 120$, and we easily find the corresponding values of (p, q, r) in each of the three cases. Finally, the solutions of our Diophantine equation are:

$$(2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 2, n) \quad \text{for} \quad n = 2, 3, \ldots.$$

The corresponding tilings of the sphere (and their Coxeter schemes) are shown in Figure 6.3.

6.7. Two-dimensional elliptic geometry

6.7.1. Spherical geometry is closely related to the *elliptic geometry* invented by Riemann. Elliptic geometry is obtained from spherical geometry by "identifying opposite points of $\mathbb{S}^2$". The precise definition can be stated as follows. Consider the set $\mathbb{E}\mathrm{ll}^2$ whose elements are pairs of antipodal points $(x, -x)$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The group $\mathrm{O}(3)$ acts on this set (because isometries of $\mathbb{S}^2$ take antipodal pairs of points to antipodal pairs), thus defining a geometry in the sense of Klein $(\mathbb{E}\mathrm{ll}^2 : \mathrm{O}(3))$, which we call *two-dimensional elliptic geometry.*

Lines in elliptic geometry are defined as great circles of the sphere $\mathbb{S}^2$, angles and distances are defined as in spherical geometry, and

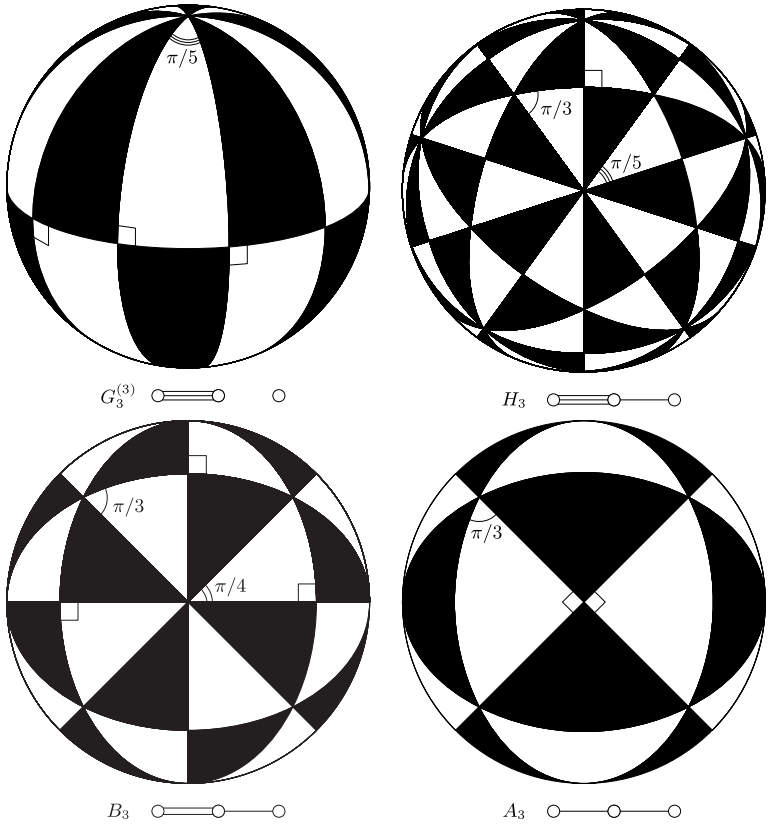


Figure 6.3. Four Coxeter tilings of the sphere.

the trigonometry of triangles in elliptic geometry is the same as in spherical geometry. More generally, one can say that elliptic geometry is locally the same as spherical, but these geometries are drastically different globally. In particular, in elliptic geometry

- *one and only line passes through any two distinct points*;

- *for a given line and any given point* (*except one, called the pole of that line*) *there exists a unique perpendicular to that line passing through the point.*

The relationship between the two geometries is best expressed by the following statement, which yields simple proofs of the statements about elliptic geometry made above.

Theorem 6.7.2. *There exists a surjective morphism*

$$D : (\mathbb{S}^2 : \mathrm{O}(3)) \to (\mathbb{E}\mathrm{ll}^2 : \mathrm{O}(3))$$

of spherical geometry onto elliptic geometry which is a local isomorphism (in the sense that any domain contained in a half-sphere is mapped bijectively and isometrically onto its image).

Proof. The map D is the obvious one: $D : x \mapsto (x, -x)$, while the homomorphism of the transformation groups is the identity isomorphism. All the assertions of the theorem are immediate. □

As we noted before, globally the two geometries are very different. Being metric spaces, they are topological spaces (in the metric topology) which are not even homeomorphic: one is a two-sided surface ($\mathbb{S}^2$), the other ($\mathbb{R}P^2$) is one-sided (it contains a Möbius strip).

6.8. Problems

In all the problems below a, b, c are the sides and α, β, γ are the opposite angles of a spherical triangle. The radius of the sphere is $R = 1$.

6.1. Prove the first cosine theorem on the sphere $\mathbb{S}^2$:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

6.2. Prove the second cosine theorem on the sphere $\mathbb{S}^2$:

$$\cos \alpha + \cos \beta \cos \gamma = \sin \beta \sin \gamma \cos a.$$

6.3. Prove that $a + b + c < 2\pi$.

6.4. Does the Pythagorean theorem hold in spherical geometry? Prove the analog of that theorem stated in Corollary 6.5.3.

6.5. Does the Moscow–New York flight fly over Spain? Over Greenland? Check your answer by stretching a thin string between Moscow and New York on a globe.

6.6. Find the infimum and the supremum of the sum of the angles of an equilateral triangle on the sphere.

6.7. The city A is located at the distance 1000km from the cities B and C; the trajectories of the flights from A to B and from A to C are perpendicular to each other. Estimate the distance between B and C. (You can take the radius of the Earth equal to 6400km.)

6.8*. Find the area of the spherical disk of radius r (i.e., the domain bounded by a spherical circle of radius r).

6.9. Find fundamental domains for the actions of the isometry groups of the tetrahedron, the cube, the dodecahedron, and the icosahedron on the 2-sphere and indicate the number of their images under the corresponding group action.

6.10. Prove that any spherical triangle has a circumscribed and an inscribed circle.

6.11. Prove that the medians of a spherical triangle intersect at one point.

6.12. Prove that the altitudes of a spherical triangle always intersect at one point.

6.13. Suppose that the medians and the altitudes of a spherical triangle interest at the points M and A, respectively. Can it happen that $M = A$?

Chapter 7

The Poincaré Disk Model of Hyperbolic Geometry

In this chapter, we begin our study of the most popular of the non-Euclidean geometries – hyperbolic geometry, concentrating on the case of dimension two. We avoid the intricacies of the axiomatic approach and define hyperbolic plane geometry via the beautiful Poincaré disk model, which is the geometry of the disk determined by the action of a certain transformation group acting on the disk (namely, the group generated by reflections in circles orthogonal to the boundary of the disk).

In order to describe the model, we need some facts from Euclidean plane geometry, which should be studied in high school, but, unfortunately, in most cases, are not. So we begin by recalling some properties of inversion (which will be the main ingredient of the transformation group of our geometry) and some constructions related to orthogonal circles in the Euclidean plane. We then establish the basic facts of hyperbolic plane geometry and finally digress, following Poincaré's argumentation from his book *Science et Hypothèse* (for the English version, see [12]) about epistomological questions relating this geometry (and other geometries) to the physical world.

7.1. Inversion and orthogonal circles

7.1.1. Inversion and its properties. The main tool that we will need in this chapter is inversion, a classical transformation from elementary plane geometry. Denote by $\mathcal{R}$ the plane $\mathbb{R}^2$ with an added extra point (called the *point at infinity* and denoted by ∞). The set $\mathcal{R} := \mathbb{R}^2 \cup \infty$ can also be interpreted as the complex numbers $\mathbb{C}$ with the "point at infinity" added; it is then called the *Riemann sphere* and denoted by $\overline{\mathbb{C}}$.

An *inversion* with center $O \in \mathbb{R}^2$ and radius $r > 0$ is the transformation of $\mathcal{R}$ that maps each point M to the point N on the ray OM so that

$$\boxed{|OM| \cdot |ON| = r^2} \tag{7.1}$$

and interchanges the points O and ∞. Sometimes inversions are called *reflections* with respect to the *circle of inversion*, i.e., the circle of radius r centered at O.

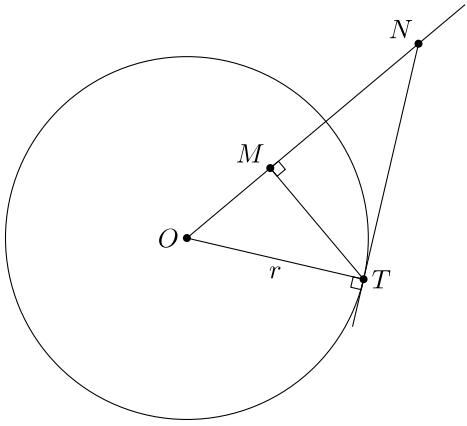


Figure 7.1. Inversion $|OM| \cdot |ON| = r^2$.

There is a simple geometric way of constructing the image of a point M under an inversion with center O and radius r: draw the circle of inversion, draw the perpendicular to OM from M to its intersection point T with the circle and construct the tangent to the

circle at T to its intersection point N with the ray OM; then N will be the image of M under the given inversion. Indeed, the two right triangles OMT and OTN are similar (they have a common acute angle at O), and therefore

$$\frac{|OM|}{|OT|} = \frac{|OT|}{|ON|},$$

and since $|OT| = r$, we obtain (7.1).

If the extended plane $\mathcal{R}$ is interpreted as the Riemann sphere $\overline{\mathbb{C}}$, then an example of an inversion (with center O and radius 1) is the map $z \mapsto 1/\overline{z}$, where the bar over z denotes complex conjugation.

It follows immediately from the definition that inversions are bijections of $\mathcal{R} = \overline{\mathbb{C}}$ that leave the points of the circle of inversion in place, "turn the circle inside out" in the sense that points inside the circle are taken to points outside it (and vice versa), and are *involutions* (i.e., the composition of an inversion with itself is the identity). Further, inversions possess the following important properties.

(i) *Inversions map any circle or straight line orthogonal to the circle of inversion into itself.* Look at Figure 7.2, which shows two orthogonal circles $\mathcal{C}_O$ and $\mathcal{C}_I$ of centers O and I, respectively.

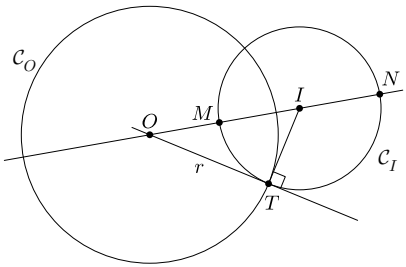


Figure 7.2. Orthogonal circles.

It follows from the definition of orthogonality that the tangent from the center O of $\mathcal{C}_O$ to the other circle $\mathcal{C}_I$ passes through the intersection point T of the two circles. Now let us consider the inversion with center O and radius $r = |OT|$. According to property (i) above, it takes the circle $\mathcal{C}_I$ to itself; in particular, the point M is

mapped to N, the point T (as well as the other intersection point of the two circles) stays in place, and the two arcs of $\mathcal{C}_I$ cut out by $\mathcal{C}_O$ are interchanged. Note further that, vice versa, the inversion in the circle $\mathcal{C}_I$ transforms $\mathcal{C}_O$ in an analogous way.

(ii) *Inversions map any circle or straight line into a circle or straight line.* In particular, lines passing through the center of inversion are mapped to themselves (but are "turned inside out" in the sense that O goes to ∞ and vice versa, while the part of the line inside the circle of inversion goes to the outside part and vice versa); circles passing through the center of inversion are taken to straight lines, while straight lines not passing through the center of inversion are taken to circles passing through that center (see Figure 7.3).

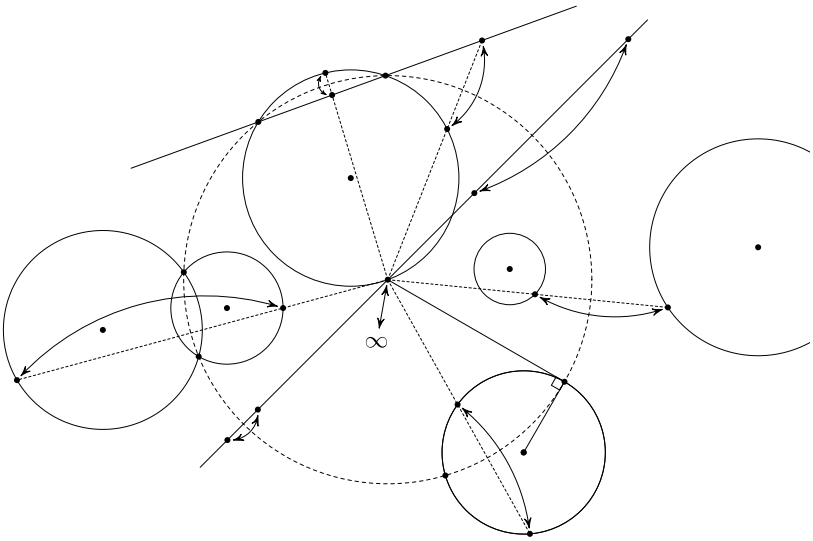


Figure 7.3. Images of circles and lines under inversion.

(iii) *Inversions preserve* (*the measure of*) *angles;* here by the measure of an angle formed by two intersecting curves we mean the ordinary (Euclidean) measure of the angle formed by their tangents at the intersection point.

The (elementary) proofs of properties (i)–(iii) are left to the reader (see Problems 7.1–7.3).

7.1.2. Construction of orthogonal circles. We have already noted the important role that orthogonal circles play in inversion (see 7.1.1(i)). Here we will describe several constructions of orthogonal circles that will be used in subsequent sections.

Lemma 7.1.3. *Let A be a point inside a circle $\mathcal{C}$ centered at some point O; then there exists a circle orthogonal to $\mathcal{C}$ such that the reflection in this circle takes A to O.*

Proof. From A draw the perpendicular to line OA to its intersection T with the circle $\mathcal{C}$ (see Figure 7.4).

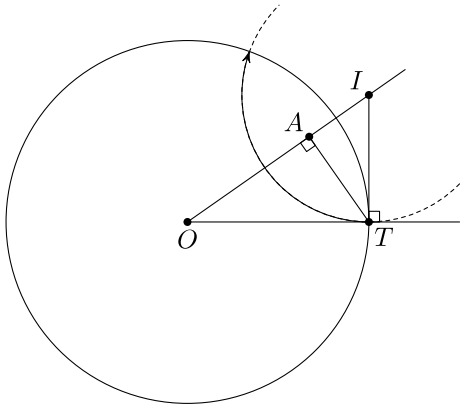


Figure 7.4. Inversion taking an arbitrary point A to O.

Draw the tangent to $\mathcal{C}$ at T to its intersection at I with OA. Then the circle of radius IT centered at I is the one we need. Indeed, the similar right triangles IAT and ITO yield $|IA|/|IT| = |IT|/|IO|$, whence we obtain $|IA| \cdot |IO| = |IT|^2$, which means that O is the reflection of A in the circle of radius $|IT|$ centered at I, as required. □

Corollary 7.1.4. (i) *Let A and B be points inside a circle $\mathcal{C}_O$ not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_O$ and passing through A and B.*

(ii) *Let A be a point inside a circle $\mathcal{C}_O$ and P a point on $\mathcal{C}_O$, with A and P not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_O$ passing through A and P.*

(iii) *Let P and Q be points on a circle $\mathcal{C}_O$ of center O such that PQ is not a diameter; then there exists a unique circle $\mathcal{C}$ orthogonal to $\mathcal{C}_O$ and passing through P and Q.*

(iv) *Let A be a point inside a circle $\mathcal{C}_O$ of center O and $\mathcal{D}$ a circle orthogonal to $\mathcal{C}_O$; then there exists a unique circle $\mathcal{C}$ orthogonal to both $\mathcal{C}_O$ and $\mathcal{D}$ and passing through A.*

Proof. To prove (i), we describe an effective step-by-step construction, which can be carried out by ruler and compass, yielding the required circle. The construction is shown in Figure 7.5, with the numbers in parentheses near each point indicating at which step the point was obtained.

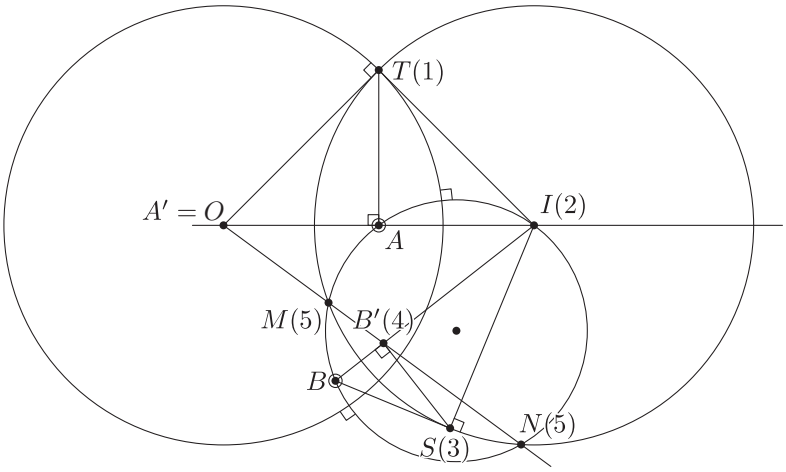


Figure 7.5. Circle orthogonal to $\mathcal{C}_O$ containing A, B.

First, we apply Lemma 7.1.3, to define an inversion φ taking A to the center O of the given circle; to do this, we draw a perpendicular from A to OA to its intersection $T(1)$ with $\mathcal{C}$, then draw the perpendicular to OT from T to its intersection $I(2)$ with OA; the required

inversion is centered at I and is of radius $|IT|$. Joining B and I, we construct the tangent $BS(3)$ to the circle of the inversion φ and find the image $B'(4)$ of B under φ by dropping a perpendicular from S to IB.

Next, we draw the line $B'O$ and obtain the intersection points M, N of this line with the circle of the inversion φ. Finally, we draw the circle $\mathcal{C}$ passing through the points M, N, I. Then $\mathcal{C}$ "miraculously" passes through A and B and is orthogonal to $\mathcal{C}_O$! Of course, there is no miracle in this: $\mathcal{C}$ passes through A and B because it is the inverse image under φ of the line OB' (see 7.1.1(ii)), and it is orthogonal to $\mathcal{C}_O$ since so is OB' (see 7.1.1(iii)).

Uniqueness is obvious in the case $A = O$ and follows in the general case by 7.1.1(ii)–(iii).

The proof of (ii) is analogous: we send A to O by an inversion φ, join O and $\varphi(P)$ and continue the argument as above.

To prove (iii), construct lines OP and OQ, draw perpendiculars to these lines from P and Q, respectively, and denote by I their intersection point. Then the circle of radius $|IP|$ centered at I is the required one. Its uniqueness is easily proved by contradiction.

To prove (iv), we again use Lemma 7.1.3 to construct an inversion φ that takes $\mathcal{C}_0$ to itself and sends A to O. From the point O, we draw the (unique) ray $\mathcal{R}$ orthogonal to $\varphi(\mathcal{L})$. Then the circle $\varphi^{-1}(\mathcal{R})$ is the required one. □

7.2. Definition of the disk model

7.2.1. The disk model of the *hyperbolic plane* is the geometry $(\mathbb{H}^2 : \mathcal{M})$ whose points are the points of the open disk

$$\mathbb{H}^2 := \big\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\big\},$$

and whose transformation group $\mathcal{M}$ is the group generated by reflections in all the circles orthogonal to the boundary circle

$$\mathbb{A} := \{(x, y) : x^2 + y^2 = 1\}$$

of $\mathbb{H}^2$, and by reflections in all the diameters of the circle $\mathbb{A}$. Now $\mathcal{M}$ is indeed a transformation group of $\mathbb{H}^2$: the discussion in 7.1.1

implies that a reflection of the type considered takes points of $\mathbb{H}^2$ to points of $\mathbb{H}^2$ and, being its own inverse, we have the implication $\varphi \in \mathcal{M} \Longrightarrow \varphi^{-1} \in \mathcal{M}$.

We will often call $\mathbb{H}^2$ the *hyperbolic plane*. The boundary circle $\mathbb{A}$ (which is not part of the hyperbolic plane) is called the *absolute*.

7.2.2. We will see later that $\mathcal{M}$ is actually the isometry group of hyperbolic geometry with respect to the *hyperbolic distance*, which will be defined in the next chapter. We will see that although the Euclidean distance between points of $\mathbb{H}^2$ is always less than 2, the hyperbolic plane is unbounded with respect to the hyperbolic distance. Endpoints of a short segment (in the Euclidean sense!) near the absolute are very far away from each other in the sense of hyperbolic distance.

Figure 7.6 gives an idea of what an isometric transformation (the simplest one – a reflection in a line) does to a picture. Note that from our Euclidean point of view, the reflection changes the size and the shape of the picture, whereas from the hyperbolic point of view, the size and shape of the image are exactly the same as those of the original. It should also be clear that *hyperbolic reflections reverse orientation*, e.g., the image of a right hand under reflection will look like a left hand, but of somewhat different size and shape.

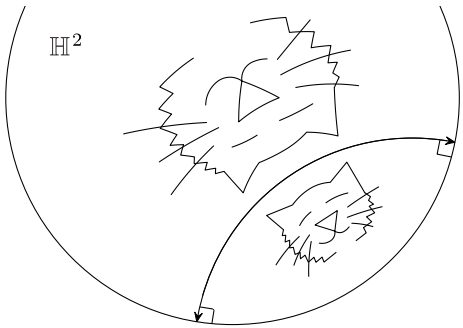


Figure 7.6. An isometry in the hyperbolic plane.

7.3. Points and lines in the hyperbolic plane

7.3.1. First we define *points of the hyperbolic plane* simply as points of the open disk $\mathbb{H}^2$. We then define the *lines* on the hyperbolic plane as the intersections with $\mathbb{H}^2$ of the (Euclidean) circles orthogonal to the absolute as well as the diameters (without endpoints) of the absolute (see Figure 7.7).

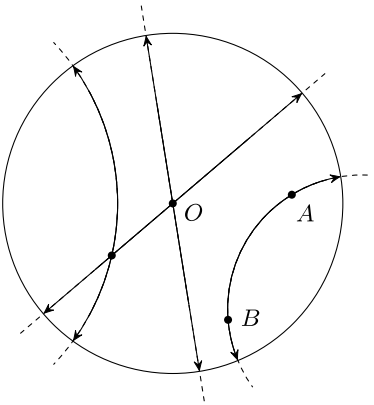


Figure 7.7. Lines on the hyperbolic plane.

Note that the endpoints of the arcs and the diameters do not belong to the hyperbolic plane: they lie in the absolute, whose points are not points of our geometry.

Thus the hyperbolic plane, as well as the lines in it, is not compact. Its compactification (or closure) is the compact disk $\overline{\mathbb{H}}^2$.

Figure 7.7 shows that some lines intersect in one point, others have no common points, and none have two common points (unlike lines in spherical geometry). This is not surprising, because we have the following statement.

Theorem 7.3.2. *One and only one line passes through any pair of distinct points of the hyperbolic plane.*

Proof. The theorem immediately follows from Corollary 7.1.4(i). □

7.4. Perpendiculars

7.4.1. Two lines in $\mathbb{H}^2$ are called *perpendicular* if they are orthogonal in the sense of elementary Euclidean geometry. When both are diameters, they are perpendicular in the usual sense, when both are arcs of circles, they have perpendicular tangents at the intersection point, and when one is an arc and the other, a diameter, then the diameter is perpendicular to the tangent to the arc at the intersection point.

Theorem 7.4.2. *There is one and only one line passing through a given point and perpendicular to a given line.*

Proof. The theorem immediately follows from Corollary 7.1.4(iv). □

7.5. Parallels and nonintersecting lines

7.5.1. Let l be a line and P a point of the hyperbolic plane $\mathbb{H}^2$ not contained in the line l. Denote by A and B the points at which l intersects the absolute. Consider the lines $k = PA$ and $m = PB$ and denote their second intersection points with the absolute by A' and B'. Clearly, the lines k and m do not intersect l. Moreover, any line passing through P between k and m (i.e., any line containing P and joining the arcs AB' and BA') does not intersect l. The lines APA' and BPB' are called *parallels* to l through P, and the lines between them are called *nonintersecting lines* with l.

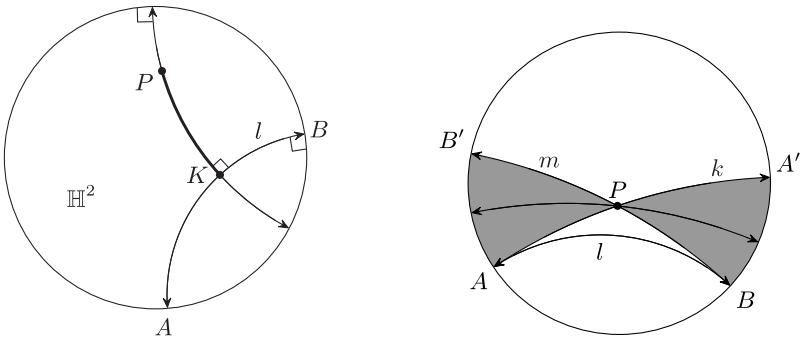


Figure 7.8. Perpendiculars and parallels.

We have proved the following statement.

Theorem 7.5.2. *There are infinitely many lines passing through a given point P not intersecting a given line l if $P \notin l$. These lines are all located between the two parallels to l.* □

This theorem contradicts Euclid's famous *Fifth Postulate*, which, in its modern formulation, says that one and only one parallel to a given line passes through a given point. For more than two thousand years, many attempts to prove that the Fifth Postulate follows from Euclid's other postulates (which, unlike the Fifth Postulate, were simple and intuitively obvious) were made by mathematicians and philosophers. Had such a proof been found, Euclidean geometry could have been declared to be an absolute truth both from the physical and the philosophical points of view; it would have been an example of a set of facts that the German philosopher Kant included in the category of *synthetic a priori*. For two thousand years, the naive belief among scientists in the absolute truth of Euclidean geometry made it difficult for the would be discoverers of other geometries to realize that they had found something worthwhile. Thus the appearance of a consistent geometry in which the Fifth Postulate does not hold was not only a crucial development in the history of mathematics, but one of the turning points in the philosophy of science. In this connection, see the discussion in Chapter 11.

7.6. Sum of the angles of a triangle

7.6.1. Consider three points A, B, C not on one line. The three segments AB, BC, CA (called *sides*) form a *triangle* with *vertices* A, B, C. The *angles* of the triangle, measured in radians, are defined as equal to the (Euclidean measure of the) angles between the tangents to the sides at the vertices.

Theorem 7.6.2. *The sum of the angles α, β, γ of a triangle ABC is less than two right angles:*

$$\boxed{\alpha + \beta + \gamma < \pi}.$$

Proof. In view of Lemma 7.1.3, we can assume, without loss of generality, that A is O (the center of $\mathbb{H}^2$). But then if we compare the

hyperbolic triangle OBC with the Euclidean triangle OBC, we see that they have the same angle at O, but the Euclidean angles at B and C are larger than their hyperbolic counterparts (look at Figure 7.9), which implies the claim of the theorem. □

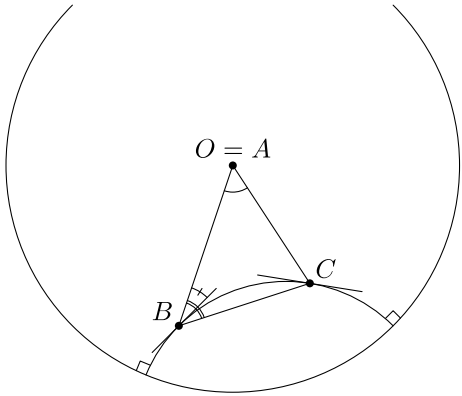


Figure 7.9. Sum of the angles of a hyperbolic triangle.

It is easy to see that very small triangles have angle sums very close to π; in fact, *the least upper bound of the angle sum of hyperbolic triangles is exactly* π. Further, *the greatest lower bound of these sums is* 0. To see this, divide the absolute into three equal arcs by three points P, Q, R and construct three circles orthogonal to the absolute passing through the pairs of points P and Q, Q and R, R and P. These circles exist by Corollary 7.1.4(iii). Then all the angles of the "triangle" PQR are zero, so its angle sum is zero. Of course, PQR is not a real triangle in our geometry (its vertices, being on the absolute, are not points of $\mathbb{H}^2$), but if we take three points P', Q', R' close enough to P, Q, R, then the angle sum of triangle $P'Q'R'$ will be less than any prescribed $\varepsilon > 0$.

7.7. Rotations and circles in the hyperbolic plane

We mentioned previously that distance between points of the hyperbolic plane will be defined later. Recall that the hyperbolic plane is

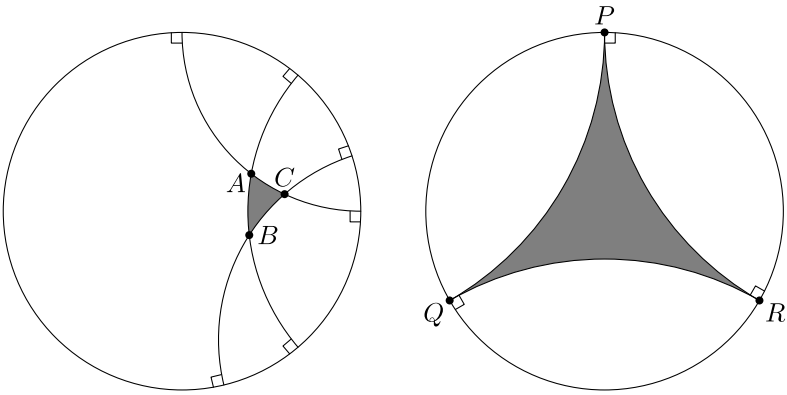


Figure 7.10. Ordinary triangle and "triangle" with angle sum 0.

the geometry $(\mathbb{H}^2 : \mathcal{M})$, in which, by definition, $\mathcal{M}$ is the transformation group generated by all reflections in all the lines of $\mathbb{H}^2$. If we take the composition of two reflections in two intersecting lines, then what we get should be a "rotation", but we can't assert that at this point, because we don't have any definition of rotation: the usual (Euclidean) definition of a rotation or even that of a circle cannot be given until distance is defined.

But the notions of rotation and of circle *can* be defined without appealing to distance in the following natural way: a *rotation* about a point $P \in \mathbb{H}^2$ is, by definition, the composition of any two reflections in lines passing through P. If I and A are distinct points of $\mathbb{H}^2$, then the (hyperbolic) *circle* of center I and radius IA is the set of images of A under all rotations about I.

Theorem 7.7.1. *A (hyperbolic) circle in the Poincaré disk model is a Euclidean circle, and vice versa, any Euclidean circle inside* $\mathbb{H}^2$ *is a hyperbolic circle in the geometry* $(\mathbb{H}^2 : \mathcal{M})$.

Proof. Let $\mathcal{C}$ be a circle of center I and radius IA in the geometry $(\mathbb{H}^2 : \mathcal{M})$. Using Lemma 7.1.3, we can send I to the center O of $\mathbb{H}^2$ by a reflection φ. Let ρ be a rotation about I determined by two lines l_1 and l_2. Then the lines $d_1 := \varphi(l_1)$ and $d_2 := \varphi(l_2)$ are diameters of the absolute and the composition of reflections in these diameters

is a Euclidean rotation about O (and simultaneously a hyperbolic one). This rotation takes the point $\varphi(A)$ to a point on the circle $\mathcal{C}'$ of center O and radius $O\varphi(A)$, which is simultaneously a hyperbolic and Euclidean circle. Now by Corollary 7.1.4(i), the inverse image $\varphi^{-1}(\mathcal{C}')$ will be a (Euclidean!) circle. But $\varphi^{-1}(\mathcal{C}')$ coincides with $\mathcal{C}$ by construction, so $\mathcal{C}$ is indeed a Euclidean circle in our model.

The proof of the converse assertion is similar and is left to the reader (see Problem 7.7). □

7.8. Hyperbolic geometry and the physical world

In his famous book *Science et Hypothèse*, Henri Poincaré describes the physics of a small "universe" and the physical theories that its inhabitants would create. The universe considered by Poincaré is Euclidean, plane (two-dimensional), and has the form of an open unit disk. Its temperature is 100° Farenheit at the center of the disk and decreases linearly to absolute zero at its boundary. The lengths of objects (including living creatures) are proportional to temperature.

How will a little flat creature endowed with reason and living in this disk describe the main physical laws of his universe? The first question he/she may ask could be: Is the world bounded or infinite? To answer this question, an expedition is organized; but as the expedition moves towards the boundary of the disk, the legs of the explorers become smaller, their steps shorter – they will never reach the boundary, and conclude that the world is infinite.

The next question may be: Does the temperature in the universe vary? Having constructed a thermometer (based on different expansion coefficients of various materials), scientists carry it around the universe and take measurements. However, since the lengths of all objects change similarly with temperature, the thermometer gives the same measurement all over the universe – the scientists conclude that the temperature is constant.

Then the scientists might study straight lines, i.e., investigate what is the shortest path between two points. They will discover that the shortest path is what we perceive to be the arc of the circle

containing the two points and orthogonal to the boundary disk (this is because such a circular path brings the investigator nearer to the center of the disk, and thus increases the length of his steps). Further, they will find that the shortest path is unique and regard such paths as "straight lines".

Continuing to develop geometry, the inhabitants of Poincaré's little flat universe will decide that there is more than one parallel to a given line passing through a given point, the sum of angles of triangles is less than π, and obtain other statements of hyperbolic geometry.

Thus they will come to the conclusion that they live in an infinite flat universe with constant temperature governed by the laws of hyperbolic geometry. But this is not true – their universe is a finite disk, its temperature is variable (tends to zero towards the boundary) and the underlying geometry is Euclidean, not hyperbolic!

The philosophical conclusion of Poincaré's argument is not agnosticism – he goes further. The physical model described above, according to Poincaré, shows not only that the truth about the universe cannot be discovered, but that it makes no sense to speak of any "truth" or approximation of truth in science – pragmatically, the inhabitants of his physical model are perfectly right to use hyperbolic geometry as the foundation of their physics because it is convenient, and it is counterproductive to search for any abstract Truth which has no practical meaning anyway.

This conclusion has been challenged by other thinkers, but we will not get involved in this philosophical discussion.

7.9. Problems

7.1. Prove that inversion maps circles and straight lines to circles or straight lines.

7.2. Prove that inversion maps any circle orthogonal to the circle of inversion into itself.

7.3. Prove that inversion is conformal (i.e., it preserves the measure of angles).

7.4. Prove that if P is a point lying outside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (also called the *power of P with respect to γ*) does not depend on the choice of l.

7.5. Prove that if P is a point lying inside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (also called the *power of P with respect to γ*) does not depend on the choice of l.

7.6. Prove that inversion with respect to a circle orthogonal to a given circle $\mathcal{C}$ maps the disk bounded by $\mathcal{C}$ bijectively onto itself.

7.7. Prove that any Euclidean circle inside the disk model is also a hyperbolic circle. Does the ordinary (Euclidean) center coincide with its "hyperbolic center"?

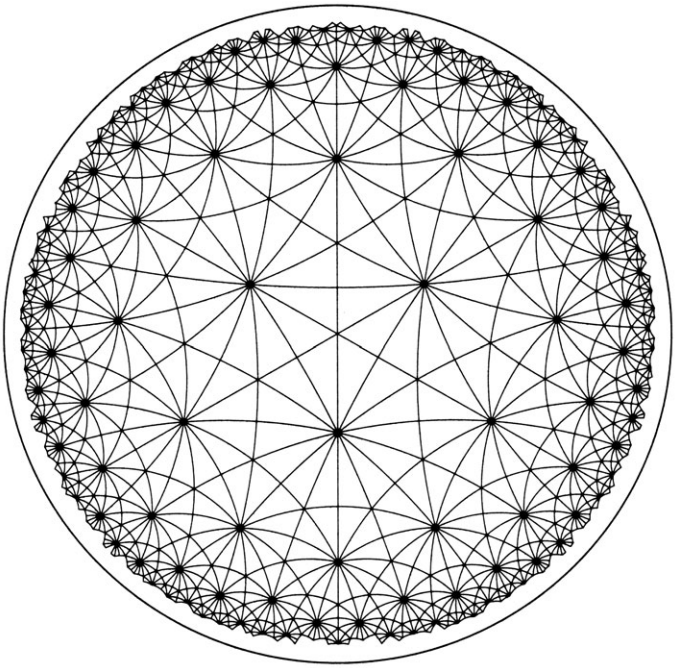


Figure 7.11. A pattern of lines in $\mathbb{H}^2$.

7.8. Study Figure 7.11. Does it demonstrate any tilings of $\mathbb{H}^2$ by regular polygons? Of how many sides? Do you discern a Coxeter geometry in this picture with "hyperbolic Coxeter triangles" as fundamental domains? What are their angles?

7.9. Prove that any inversion of $\overline{\mathbb{C}}$ preserves the cross ratio of four points:

$$\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

7.10*. Using complex numbers, invent a formula for the distance between points on the Poincaré disk model and prove that "symmetry with respect to straight lines" (i.e., inversion) preserves this distance.

7.11. Prove that hyperbolic geometry is homogeneous in the sense that for any two flags (i.e., half-planes with a marked point on the boundary) there exists an isometry taking one flag to the other.

7.12. Prove that the hyperbolic plane (as defined via the Poincaré disk model) can be tiled by regular pentagons.

7.13. Define inversion (together with the center and the sphere of inversion) in Euclidean space $\mathbb{R}^3$, state and prove its main properties: inversion takes planes and spheres to planes or spheres, any sphere orthogonal to the sphere of inversion to itself, any plane passing through the center of inversion to itself.

7.14. Using the previous problem, prove that any inversion in $\mathbb{R}^3$ takes circles and straight lines to circles or straight lines.

7.15. Prove that any inversion in $\mathbb{R}^3$ is conformal (preserves the measure of angles).

7.16. Construct a model of hyperbolic space geometry on the open unit ball (use Problem 7.13).

7.17. Prove that there is a unique common perpendicular joining any two nonintersecting lines.

7.18. Let $A_\infty P$ and $A_\infty P'$ be two parallel lines (with A_∞ a point on the absolute). Given a point M on $A_\infty P$, we say that $M' \in A_\infty P'$ is

the *corresponding point* to M if the angles $A_\infty MM'$ and $A_\infty M'M$ are equal. Prove that any point $M \in A_\infty P$ has a unique corresponding point on the line $A_\infty P'$.

7.19. The locus of all points corresponding to a point M on $A_\infty P$ and lying on all the parallels to $A_\infty P$ is known as a *horocycle*. What do horocycles look like in the Poincaré disk model?

Chapter 8

The Poincaré Half-Plane Model

In this chapter, we will present another model of the hyperbolic plane, also due to Poincaré. This model is also a geometry in the sense of Klein, and we will learn in subsequent chapters that it is actually isomorphic (as a geometry) to the disk model studied in Chapter 7.

The points of the half-plane model are simply complex numbers with positive imaginary part (the part of the complex numbers that lies "above" the real axis). Such a configuration of points does not appear to be as symmetric as that of the disk, but the half-plane model has the advantage that the elements of its transformation group (which is a concrete subgroup of the Möbius group of linear fractional transformations, see the definition below) are defined by simple explicit formulas and there is a neat formula for the distance between two points.

It will turn out that the isometry group with respect to this distance is actually the transformation group of the model, so that this model shows that hyperbolic geometry is a geometry in the traditional sense: its structure is defined by a distance function. This will allow us to study "hyperbolic trigonometry", and understand the meaning of certain mysterious "absolute constants" that arise in hyperbolic plane geometry.

In order to define the half-plane model, we will need to specify certain transformation groups acting on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$, and we begin this chapter by studying these transformations.

8.1. Affine and linear-fractional transformations of $\overline{\mathbb{C}}$

In this section, we will be studying various linear-fractional groups acting on the Riemann sphere $\overline{\mathbb{C}}$. An efficient tool in our constructions will be the notion of cross ratio, with which we begin.

8.1.1. Cross ratio of four complex numbers. The *cross ratio* of four complex numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}$ is defined as the number

$$\boxed{\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}}. \tag{8.1}$$

8.1.2. Affine transformations. A transformation of $\overline{\mathbb{C}}$ onto itself of the form $z \mapsto az + b$, $\infty \mapsto \infty$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called *affine*. In particular, if $a = 1$, the corresponding affine transformation is a parallel translation (by the vector OB, where B is the point of the complex plane corresponding to the complex number b).

Theorem 8.1.3. *Affine transformations take straight lines to straight lines, circles to circles, and preserve angles and cross ratios.*

Proof. Denoting $a = re^{i\varphi}$, $r > 0$, we can write

$$z \mapsto e^{i\varphi} z \mapsto r(e^{i\varphi} z) \mapsto (re^{i\varphi} z) + b = az + b,$$

which shows that any affine transformation is the composition of a rotation (by the angle φ), a homothety (with coefficient r), and a parallel translation (by the vector b). This implies the theorem, because rotations, homotheties, and translations obviously possess all four of the properties asserted by the theorem. The least obvious of these facts is that homotheties preserve cross ratio, but this follows immediately from the fact that homothety in the plane of the complex variable is multiplication by a real number (which will cancel out in each of the fractions of the cross ratio). □

8.1.4. Linear-fractional transformations. A transformation of $\overline{\mathbb{C}}$ given on $\mathbb{C} \setminus \{-d/c\}$ by

$$\boxed{z \mapsto \frac{az+b}{cz+d}\,, \quad \text{where} \quad cb-ad \neq 0}, \tag{8.2}$$

which takes the point $-d/c$ to ∞ and ∞ to a/c, is called *linear-fractional*.

The set of all linear-fractional transformations forms a group, called the *Möbius group* and denoted by Möb.

Indeed, the fact that the composition of two linear-fractional transformations is a linear-fractional transformation can be shown as follows: substitute $(a_1z+b_1)/(c_1z+d_1)$ for z in the expression $(az+b)/(cz+d)$, which yields (after some manipulations)

$$\frac{(aa_1+bc_1)z+(ab_1+bd_1)}{(ca_1+dc_1)z+(cb_1+dd_1)}; \tag{8.3}$$

but this expression is of the same form as (8.2), so the composition is indeed linear-fractional.

The fact that the inverse of any linear-fractional transformation is a linear-fractional transformation is also easy to prove. To do that, it suffices to find values of a_1, b_1, c_1, d_1 (in terms of a, b, c, d) so that the expression (8.3) reduces to $(1\cdot z+0)/(0\cdot z+1)$; such values must satisfy the system of four linear equations in four unknowns,

$$aa_1+bc_1=1, \quad ab_1+bd_1=0, \quad ca_1+dc_1=0, \quad cb_1+dd_1=1,$$

but this system obviously has a nonzero solution.

The following property of linear-fractional transformations gives an insight in the geometric meaning of this class of transformations and turns out to be extremely useful in constructing and analyzing them.

Lemma 8.1.5. *Let z_1, z_2, z_3 and w_1, w_2, w_3 be two triplets of distinct points of the Riemann sphere. Then there exists a unique linear-fractional transformation taking z_i to w_i, $i = 1, 2, 3$.*

Theorem 8.1.6. *Linear-fractional transformations take straight lines and circles to straight lines or circles, and preserve angles and cross ratios.*

Proof. As can be easily checked, the image of the point z under the linear-fractional transformation (8.1) may be rewritten as

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)},$$

and therefore can be regarded as the composition

$$z \mapsto cz+d =: z_1 \mapsto cz_1 =: z_2 \mapsto 1/z_2 =: z_3 \mapsto (bc-ad)z_3 =: z_4$$
$$\mapsto \frac{a}{c} + z_4 = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = \frac{az+b}{cz+d}$$

of an affine transformation, a homothety, a transformation taking z to $1/z$, another homothety, and a parallel translation. Concerning all of these transformations, except $z \mapsto 1/z$, we know that they take straight lines to straight lines, circles to circles, and preserve angles and cross ratios.

Concerning the transformation $z \mapsto 1/z$, a straightforward if somewhat tedious calculation shows that it preserves cross ratios (one replaces z_i by $1/z_i$, $i = 1, 2, 3, 4$, and the obtained rather cumbersome fractions, after cancellations, reacquire the exact form of the original ratio). Further, since $\overline{1/z} = 1/\overline{z}$, the transformation $z \mapsto 1/z$ is the composition of a reflection, an inversion, and another reflection. But we know that inversion takes straight lines or circles to straight lines or circles and preserves angles (see 7.1.1(i)–(iii)), which proves the theorem. □

8.1.7. Two examples of linear-fractional transformations. Linear-fractional transformations are the subject matter of an important chapter of the theory of a complex variable; in it, one studies what types of domains can be mapped into each other by such transformations. We will not need the general theory of this study, but the following two examples of linear-fractional transformations will be very important for what follows.

Example 8.1. *The linear-fractional transformation*

$$\Omega : z \mapsto i \cdot \frac{1+z}{1-z}$$

maps the unit disk $\mathbb{D}^2 := \{z \in \mathbb{C} : |z|^2 \le 1\}$ *to the upper half-plane* $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Indeed, it is easy to verify that the

points $-1, i, 1$ are mapped to $0, -1, \infty$, respectively, which means (by Theorem 8.1.3) that the boundary circle of the disk $\mathbb{D}^2$ is mapped to the real axis. A simple computation shows that $|z| < 1$ implies that $\mathrm{Im}(\Omega(z)) > 0$, as required.

Example 8.2. *The linear-fractional transformations*

$$z \mapsto \frac{az+b}{cz+d} \quad \text{and} \quad z \mapsto \frac{a(-\overline{z})+b}{c(-\overline{z})+d}, \tag{8.4}$$

where $a, b, c, d \in \mathbb{R}$ and $bc - ad > 0$, take the upper half-plane to itself, the first of them preserving, the second, reversing the orientation of the half-plane.

For the first of these formulas, it is obvious that points of the real axis are taken to points of the real axis; further, if z, $\mathrm{Im}\, z > 0$, is any point in the upper half-plane, then

$$\mathrm{Im}\,\frac{az+b}{cz+d} = \mathrm{Im}\,\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \mathrm{Im}\,\frac{adz+bc\overline{z}}{|cz+d|^2} = \frac{(ad-bc)\mathrm{Im}\, z}{|cz+d|^2},$$

which is positive iff $bc - ad > 0$.

The second formula differs from the first by a transformation of the form $z \mapsto -\overline{z}$, which obviously takes the upper half-plane to itself, but reverses the orientation.

The set of all linear-fractional transformations (8.4) *constitutes a group under composition*, which we denote by $\mathbb{R}$Möb. Indeed, this follows from the fact that the set of all linear-fractional transformations of the form (8.2) is a group and a composition of transformations taking the half-plane to itself. The group $\mathbb{R}$Möb will be the transformation group of the half-plane model.

8.2. The Poincaré half-plane model

The *Poincaré half-plane model* is the geometry consisting of the points $z \in \mathbb{C}$ such that $\mathrm{Im}\, z > 0$, supplied with the transformation group $\mathbb{R}$Möb. In this geometry, *straight lines* are defined either as open half-circles (in the upper half-plane) perpendicular to the line $\mathrm{Im}\, z = 0$ (which is called the *absolute*) or as the open rays

$$\{z \in \mathbb{C} : \mathrm{Re}\, z \,=\, x_0 \in \mathbb{R}, \mathrm{Im}\, z > 0\}.$$

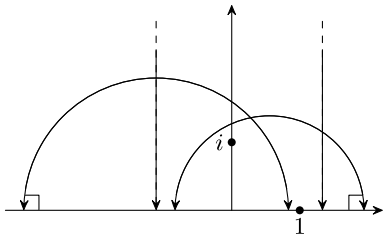


Figure 8.1. "Straight lines" in the half-plane model.

8.3. Perpendiculars and parallels

The situation with perpendiculars and parallels in the half-plane model is quite similar to that for the disk model, except that the corresponding pictures look very different.

Theorem 8.3.1. *Given a point P and a line l in the half-plane model, there exists a unique perpendicular to l passing through P.*

Proof. There are two cases to consider (depending on whether l is a half-circle or a half-line); see Figure 8.2.

In the first case, there are two different subcases: in the first subcase, the given point P' lies on the vertical line passing through the center of the half-circle l, and the foot of the desired perpendicular is obviously the intersection point K' of that vertical line with the half-circle l; in the second subcase, when the given point P is not on that vertical line, the construction of the foot K of the perpendicular reduces to a nice problem in the Euclidean geometry of circles: to find a circle orthogonal to a given one, centered on a given diameter of the given circle and passing through a given point; this problem is left to the reader as Problem 8.2.

In the second case, the construction is obvious: one considers the circle passing through P and centered at the intersection point of l and the absolute and, for the desired perpendicular, one takes the arc PK, where K is the intersection point of that circle with l. □

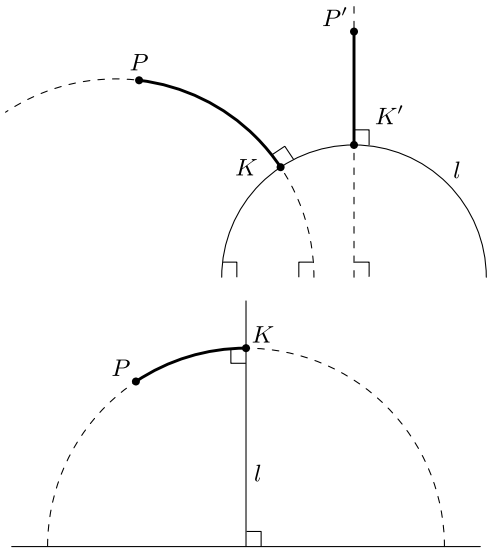


Figure 8.2. Perpendiculars in the half-plane model.

Theorem 8.3.2. *Given a point P and a line l in the half-plane model, there exist infinitely many lines passing through P and not intersecting l. All these nonintersecting lines lie between the two parallels to l from P.*

Proof. There are two cases to consider (depending on whether l is a half-line or a half-circle); see Figure 8.3.

In the first case, the two parallels are easily constructed as follows: each of them is a half-circle centered on the absolute, passing through the given point P and through one of the two intersection points of the half-circle l with the absolute. Their uniqueness is obvious.

In the second, the construction of the parallels is even simpler: for one of them, we must take the straight line passing through P and perpendicular to the absolute and for the other, the circle centered on the absolute, tangent to l, and passing through P. Uniqueness is also obvious in this case. □

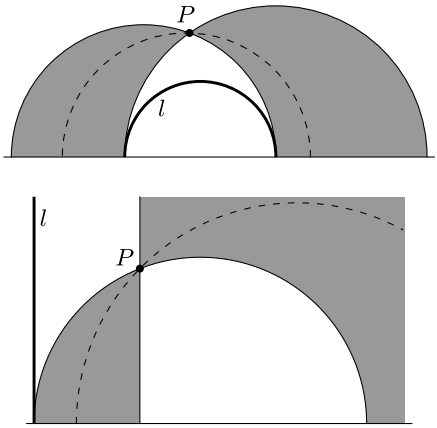


Figure 8.3. Parallels in the half-plane model.

8.4. Isometries w.r.t. Möbius distance

Let us define the *Möbius distance* $\mu(A, B)$ between two points A, B of the upper half-plane by setting

$$\boxed{\mu(A,B) := |\log(|\langle A, B, X, Y\rangle|)|},$$

where X and Y are the intersection points of the line (AB) with the absolute if the points A, B have different real parts (note that $\langle A, B, X, Y\rangle \in \mathbb{R}$ because the four points lie on a circle, so that the natural logarithm is well defined); if $\mathrm{Re}(A) = \mathrm{Re}(B) = x_0$, we set

$$\boxed{\mu(A,B) := |\ln(\langle A, B, \infty, X\rangle)|},$$

where X is the point with coordinates $(x_0, 0)$.

Theorem 8.4.1. *The isometry group of the upper half-plane with respect to the distance μ coincides with the group* $\mathbb{R}$Möb *described in Example* 8.1.8.

The proof is a tedious verification that we omit. □

8.5. Problems

8.1. Prove that

(a) linear-fractional transformations preserve the cross ratio of four points on the Riemann sphere $\overline{\mathbb{C}}$;

(b) a linear-fractional transformation is uniquely determined by three points and their images.

8.2. Let l be a straight line in the Euclidean plane, γ a circle with center O on l, P a point not on l and not on the perpendicular to l from O. Prove that there exists a unique circle passing through P, orthogonal to γ, and centered on l.

8.3. Let l be a straight line in the Euclidean plane, γ a circle with diameter AB on l, P a point not on l and not in γ. Prove that there exists a unique circle passing through P and A with center on l, and a unique circle passing through P and B with center on l.

8.4. Prove that all motions (i.e., orientation-preserving isometries) of the Poincaré disk model are of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}},$$

where a and b are complex numbers such that $|a|^2 = |b|^2 = 1$.

8.5. Show that there exists an isometry of the half-plane model that takes any flag to any other flag (a flag is a triple consisting of a line in the hyperbolic plane, one of the two half-planes that the line bounds, and a point on that line).

8.6*. Find a formula for the area of a triangle in hyperbolic geometry.

Chapter 9

The Cayley–Klein Model

In this chapter, we study one more model of hyperbolic plane geometry – the Cayley–Klein model. Its set of points consists of all the points of the open disk (just as in the case of the Poincaré disk model) and its transformation group is isomorphic to $\mathcal{M}$ (the transformation group of the Poincaré model), but the action of $\mathcal{M}$ in the two models is not the same. As a result, the lines in the two models look very different: instead of arcs of circles as in the Poincaré model, in the second model lines are open chords of the disk.

Another essential difference between our study of the two models is in the approach to the definition of the Cayley–Klein model as a geometry (in the sense of Klein), i.e., the definition of its transformation group. This is done in a more traditional way: we will begin by defining the distance between points and then introduce the transformation group of the geometry as the isometry group of this distance, i.e., the group of all distance-preserving bijections of its set of points.

9.1. Isometry and the Cayley–Klein model

9.1.1. The distance function. Let $\mathbb{H}^2$ be the interior of the unit disk on the Euclidean plane and let A and B be points of $\mathbb{H}^2$. Suppose the (Euclidean) line AB intersects the boundary of the disk $\overline{\mathbb{H}}^2$ at the

points X and Y, the points Y, A, B, X appearing on the line AB in that order (see Figure 9.1).

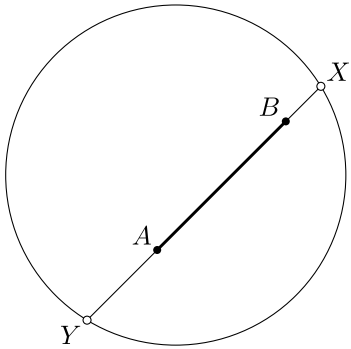


Figure 9.1. Line in the Cayley–Klein model.

Then the *distance* d between the points A and B is defined as

$$\boxed{d(A,B) := \frac{1}{2}\left|\log\left(\frac{|AX|}{|BX|} : \frac{|AY|}{|BY|}\right)\right|}. \tag{9.1}$$

The coefficient $1/2$ in the right-hand side of (9.1) can be replaced by any other positive real number c – all such distances define the same geometry (up to isomorphism, but not up to isometry). The reason for this strange choice ($c = 1/2$ rather than the more natural $c = 1$) is that the coefficient $c = 1/2$ leads to more elegant formulas than $c = 1$ and gives a metric compatible with the one in the Poincaré model.

Note that if the points Y, A, B, X are ordered on the line AB as shown in the figure (and $A \neq B$), then the expression under the logarithm sign is greater than 1 and therefore the distance between A and B is positive. Note further that if we introduce coordinates on the line AB, placing the origin "to the left" of Y and assigning the real numbers y, a, b, x to the points Y, A, B, X, respectively, then the expression under the logarithm sign can be rewritten as the following cross ratio:

$$\frac{x-a}{x-b} : \frac{y-a}{y-b} = \langle a, b, x, y\rangle.$$

This cross ratio looks very similar to the one we used to define the distance in the half-plane model, but it should be stressed that here we are dealing with real numbers rather than complex ones.

9.1.2. Properties of the distance function. *The distance function d given by* (9.1) *defines a metric on the open disk $\mathbb{H}^2$, i.e.,*

(i) $d(A,B) \geq 0$, *and* $d(A,B) = 0$ *if and only if* $A = B$;

(ii) $d(A,B) = d(B,A)$;

(iii) $d(A,B) + d(B,C) \geq d(A,C)$.

Proof. Item (i) obviously holds: the distance $d(A,B)$ between distinct points A and B is positive (as we have shown above), while if $A = B$, then the denominators in (9.1) cancel, leaving us with $\log(1) = 0$.

Item (ii) follows from the obvious formula

$$\frac{x-a}{x-b} : \frac{y-a}{y-b} = \Big(\frac{x-b}{x-a} : \frac{y-b}{y-a}\Big)^{-1}.$$

Finally, item (iii) can be proved by using projective transformations. Since we won't be using (iii) in what follows, we postpone its proof to Chapter 12 (see Problem 12.13). □

9.1.3. Definition of the Cayley–Klein model. As explained above, we will define the geometry (in the sense of definition 1.4.1) of the Cayley–Klein model by taking for its transformation group the isometry group of the distance d, i.e., the group of all distance-preserving bijections of $\mathbb{H}^2$, which we denote by $\mathcal{N}$. (We will prove later that $\mathcal{N}$ is isomorphic to $\mathcal{M}$, the transformation group of the Poincaré disk model, but this fact does not concern us now.)

Thus we define the *Cayley–Klein model* of the hyperbolic plane as the geometry $\big(\mathbb{H}^2 : \mathcal{N}\big)$, where $\mathcal{N}$ is the isometry group of the open unit disk $\mathbb{H}^2$ with respect to the distance (9.1).

9.1.4. Lines and points in the Cayley–Klein model. The *points* of the Cayley–Klein model, as explained above, are simply the points of the open unit disk $\mathbb{H}^2$ in $\mathbb{R}^2$. The boundary of the disk is traditionally called the *absolute*, and its points do *not* belong to our geometry.

The *lines* of our geometry are defined as the chords of the absolute (without their endpoints). This definition immediately implies the fundamental facts that *one and only one line passes through any two distinct points* and that *two noncoinciding lines either don't intersect or have exactly one common point.*

In the following two sections, just as in the corresponding sections in the previous two chapters, we shall derive the basic facts of hyperbolic geometry in the case of the model under consideration.

9.2. Parallels in the Cayley–Klein model

The situation with parallelism in this model is similar to that in the Poincaré disk model, except that the picture looks slightly different (rectilinear chords instead of arcs of circles).

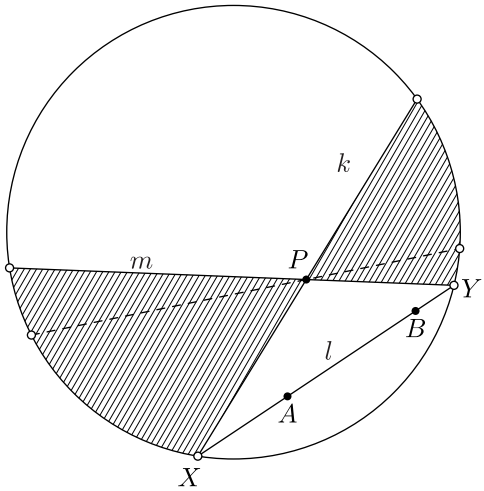


Figure 9.2. Parallels and nonintersecting lines.

9.2.1. Definitions. Given a line $l = AB$ and a point P not on this line, it is easy to describe the lines that pass through P and do not intersect l. Indeed, denoting by k and m the lines passing through P and through the intersection points X, Y of the line l with the

absolute, we see that any line passing through P and lying between k and m does not intersect line l; these lines are called *nonintersecting* lines w.r.t. AB, while the lines k and m are the *parallels* to AB passing through P (see Figure 9.2).

More generally, two lines (i.e., open chords of the disk) are *parallel* if they have no common points in $\mathbb{H}^2$ and one common point on the absolute; if two lines (chords) have no common points at all (in the closed disk $\overline{\mathbb{H}^2}$), then they are called *nonintersecting*.

We have shown that *there are infinitely many lines passing through a given point P not intersecting a given line $l = AB$ if $P \notin l$; these lines are all located between the two parallels to l passing through P.*

9.2.2. Remark. Note that the set of all lines passing through a fixed point of the absolute fills the entire hyperbolic plane $\mathbb{H}^2$ (see Figure 9.3, where both disk models are pictured).

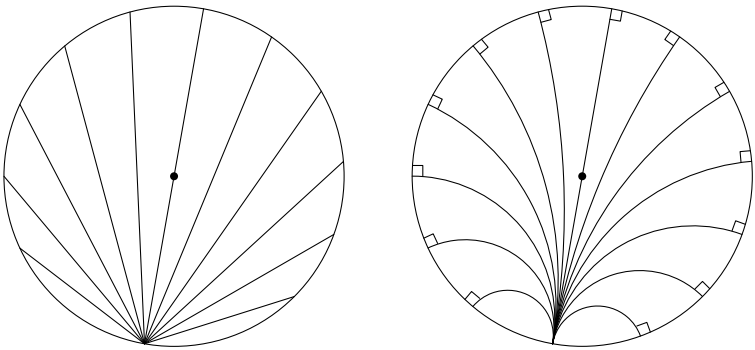


Figure 9.3. Parallels filling the hyperbolic plane.

This means that, by using the metric on each of these lines, we can try to define the notion of "parallel translation" and therefore that of a "free vector" of sorts in hyperbolic geometry. This might lead one to think that one can associate a linear space with our geometry. Unfortunately, this is not the case (see the discussion in 9.4.1 and in Problem 9.7).

9.3. Perpendiculars in the Cayley–Klein model

9.3.1. What they look like. Unlike perpendiculars in the Poincaré disk model, perpendicular lines in the Cayley–Klein model do not form right angles in the Euclidean sense. An exactly constructed example is shown in Figure 9.4 (the nontrivial geometric construction by means of which this "hyperbolically perpendicular straight line" was drawn does not appear in the figure, and will be discussed in the next chapter, in Subsection 10.1.5, and shown in Figure 10.4).

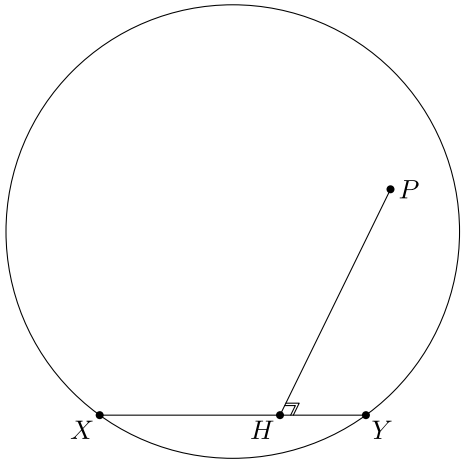


Figure 9.4. Strange looking perpendicular.

9.3.2. Definitions. Before discussing perpendicularity, we must *define* what perpendicular lines are. To do that, we first define a *reflection with respect to a given line* as the nonidentical isometry of $\mathbb{H}^2$ that takes each point of the given line to itself. Now we can define two lines as *perpendicular* if the reflection with respect to one of them takes the other line to itself. It is true that *there exists one and only one perpendicular to a given line passing through a given point*, but the proof of this fact directly in the Cayley–Klein model is quite difficult and is omitted.

9.3.3. Remark. It should be stressed that in our model the "hyperbolic measure" of angles is in general *not* equal to their Euclidean measure. In particular, triangles in the Cayley–Klein model, which look like rectilinear Euclidean triangles, have angle sums *less* than π (although visually this does not seem to be the case).

9.4. The hyperbolic line and relativity

In this section, we digress about the distance function on hyperbolic straight lines and point out a remarkable relationship between the composition of shifts on such a line and the additivity of velocities in the Special Relativity Theory of Einstein. But we begin with a general remark concerning vectors in hyperbolic geometry.

9.4.1. Remark about free vectors. The notion of free vector in Euclidean geometry, defined as an equivalence class of equal attached vectors, allows us to associate to the Euclidean plane a two-dimensional real vector space whose elements are precisely the free vectors of the Euclidean plane. Any free vector also defines parallel shifts of the entire plane in a natural way. All this is possible because at each point of the Euclidean plane there is one and only one (attached) vector pointing in the same direction and having the same length as a given (attached) vector. On the hyperbolic plane supplied with a metric, we can say when two vectors have the same length, but the expression "point in the same direction" is meaningless (compare with Remark 9.2.2), so that *there is no well-defined notion of parallel shift*. However, the notion of parallel shift *along a fixed hyperbolic straight line* makes sense, and we discuss it in the next subsection.

9.4.2. Adding shifts and velocities. Let us distinguish some hyperbolic straight line in the Cayley–Klein model (i.e., an open chord of the open disk $\mathbb{H}^2$) and parametrize it by an appropriate Euclidean parameter x so that it is isometric to the open interval $(-1, 1)$. Let v be a real number of absolute value less than 1. Consider the map

$$T_v : [-1, 1] \to [-1, 1], \quad x \mapsto \frac{x + v}{xv + 1}.$$

It is easy to prove that T_v is a bijection of the closed interval $[-1, 1]$ to itself leaving its endpoints in place and its restriction to the open

interval $(-1,1)$ is an isometry with respect to the hyperbolic distance (for the details, see Problem 9.9). This isometry can therefore be regarded as *the parallel shift along the given hyperbolic line by the vector* v.

Let us calculate the composition of two parallel shifts by the vectors v_1 and v_2:

$$\begin{aligned} x \mapsto \frac{x+v_1}{xv_1+1} &\mapsto \Big(\frac{x+v_2}{xv_2+1}+v_1\Big)\Big/\Big(\frac{x+v_2}{xv_2+1}v_1+1\Big) \\ &= \Big(x+\frac{v_1+v_2}{1+v_1v_2}\Big)\Big/\Big(x\frac{v_1+v_2}{1+v_1v_2}+1\Big); \end{aligned}$$

we see that the composition $T_{v_2}\circ T_{v_1}$ is exactly the parallel shift T_v, where v is defined by the formula

$$v := \frac{v_1+v_2}{1+v_1v_2}. \tag{9.1}$$

Thus we have proved that *the composition of two parallel shifts by vectors* v_1 *and* v_2 *is a parallel shift by the vector* v *given by formula* (9.1).

The reader will surely have noticed that this formula is the analog of the famous Einstein formula for the addition of velocities:

$$v := \frac{v_1+v_2}{(1+v_1v_2)/c^2},$$

where c is the speed of light. The two formulas differ only in the choice of the scale of velocity, and if in our hyperbolic scale the "speed of light" is set equal to 1, they coincide. Note that in both situations, if the "velocity vectors" v_1 and v_2 are very small as compared to the constant c (or 1 in our case), then v is approximately equal to v_1+v_2.

The above observation shows the deep relationship existing between special relativity and hyperbolic geometry. Is our universe hyperbolic rather than Euclidean? Actually, most physicists believe it is neither.

9.5. Problems

9.1. Prove that for three points A, B, C on one line, where B is between A and C, one has $d(A,B)+d(B,C)=d(A,C)$.

9.2. Prove that the equality $d(A,B) + d(B,C) = d(A,C)$ implies that the points A, B, C lie on one line and B is between A and C.

9.3. Prove that the reflection in a line in the Cayley-Klein model is an involution.

9.4. Show that the notion of perpendicular lines in the Cayley–Klein model (as introduced in 9.3.2) is well defined (i.e., does not depend on the order of the two lines).

9.5. Prove that the four angles formed at the intersection point of two perpendiculars are congruent.

9.6*. Prove that the sum of angles of a triangle in the Cayley–Klein model is less than π directly from the definitions pertaining to the model.

9.7. Having defined the notion of free vector in hyperbolic geometry as suggested in 9.2.2, try to define the sum of two vectors and investigate the possibility of associating a two-dimensional vector space with hyperbolic plane geometry.

9.8. Construct a triangle in the Cayley–Klein model with angle sum less than a given positive ε.

9.9. Prove that the parallel shift T_v defined in 9.4.2 does take $(-1, 1)$ to itself and find the appropriate hyperbolic distance for which it is an isometry.

Chapter 10

Hyperbolic Trigonometry and Absolute Constants

We begin this chapter by showing that the three models of the hyperbolic plane are, in fact, isomorphic geometries. In continuing and concluding our study of hyperbolic plane geometry, we will then feel free to use whichever model is more convenient in the given context. This study includes the main formulas of hyperbolic trigonometry, which we obtain after having recalled the definitions of the hyperbolic functions, usually studied in complex analysis. In conclusion of the chapter, we learn that in hyperbolic geometry, unlike Euclidean geometry, there are inherent absolute constants.

10.1. Isomorphism between the two disk models

As we mentioned in the previous chapter, the Cayley–Klein model and the Poincaré disk model are isomorphic. This means that there is a bijection between their sets of points and an isomorphism of their transformation groups which are compatible in the sense specified in 1.4.4. To prove this, we will need a classical construction from Euclidean space geometry.

10.1.1. Stereographic projection. Let $\mathbb{S}^2$ be the unit sphere, let Π be the equatorial plane of the sphere, and N its north pole. The *stereographic projection* $\sigma : \Pi \to \mathbb{S}^2$ is the map that takes each point $M \in \mathbb{S}^2 \setminus N$ to the intersection point M' of the ray NM with Π.

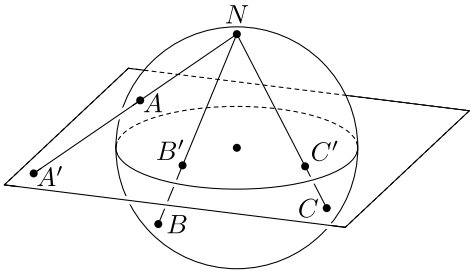


Figure 10.1. Stereographic projection.

Obviously, σ is a bijection of $\mathbb{S}^2 \setminus \{N\}$ onto Π. It is also not hard to prove that *stereographic projection is conformal* (see Problem 10.1).

10.1.2. Bijection between the sets of points of the two disk models. We regard the intersection of the open unit ball with the equatorial plane Π as the set $\mathbb{H}^2$ of points of both disk models. In order to prove that the two models are isomorphic, we begin by establishing a bijection β between their point sets. This bijection is *not the identity map*, and can be described as follows.

Let A be an arbitrary point of $\mathbb{H}^2$ and let XY be the chord (of the absolute) perpendicular to the radius OA (Figure 10.2). Consider the vertical plane containing XY; it intersects the unit sphere along a circle. Denote by A_1 the intersection of the downward vertical ray passing through A with this circle. Now join the points A_1 and N and denote by A' the intersection of A_1N and the equatorial plane. The correspondence $A \mapsto A'$ defines a map from $\mathbb{H}^2$ to $\mathbb{H}^2$ that we denote by β.

It is not hard to prove that *the map β is a bijection of $\mathbb{H}^2$ onto itself* (for the details, see Problem 10.2).

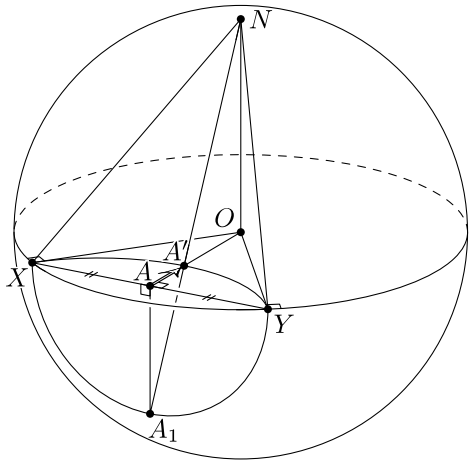


Figure 10.2. Bijection between the two disk models.

10.1.3. Isomorphism between the transformation groups. The next step in the proof of the fact that the two disk models are isomorphic geometries is the construction of an isomorphism between their transformation groups $\mathcal{N}$ and $\mathcal{M}$ that would be compatible with β. But that construction is in a sense automatic, because, as we shall see, the compatibility condition actually prescribes the choice of isomorphism.

Our aim is to construct an isomorphism $\varphi : \mathcal{N} \to \mathcal{M}$, where $\mathcal{N}$ and $\mathcal{M}$ are the transformation groups of the Cayley–Klein and the Poincaré disk models, respectively. Let $g \in \mathcal{N}$ be an arbitrary element and A an arbitrary point of the Poincaré disk. We define the element $\varphi(g)$ by setting

$$\big(\varphi(g)\big)(A) := \beta\big(g\big(\beta^{-1}(A)\big)\big),$$

where β is the bijection defined in the previous subsection. This formula says that in order to obtain the image $B := \big(\varphi(g)\big)(A)$ under $\varphi(g)$ of an arbitrary point A, we perform the only possible natural actions: pull back the point A from the Poincaré disk model to the Cayley–Klein disk via β^{-1}, obtaining $A' := \beta^{-1}(A)$, act on A' by g,

and return the obtained point $g\big(\beta^{-1}(A)\big)$ to the Poincaré disk via β (look at Figure 10.3).

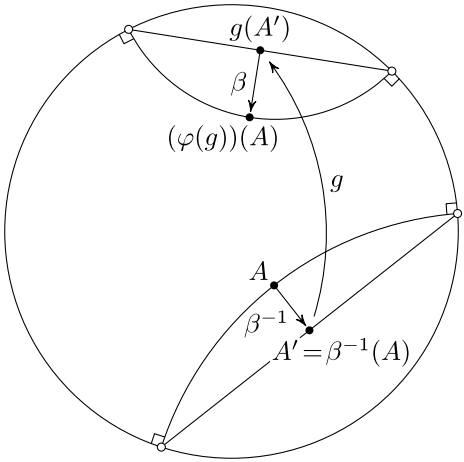


Figure 10.3. Isomorphism of the two disk models.

The fact that φ is a group homomorphism is obvious by construction, the fact that it is bijective is also easy to prove (see Problem 10.3), while the fact that the pair (β, φ) is an isomorphism of geometries is also immediate from the construction. We have proved the following theorem.

Theorem 10.1.4. *The map β from* 10.1.3 *defines an isomorphism of the geometry $(\mathbb{H}^2 : \mathcal{N})$ (the Cayley–Klein model) and the geometry $(\mathbb{H}^2 : \mathcal{M})$ (the Poincaré disk model) if we define the corresponding isomorphism (which we denote by φ) of the groups $\mathcal{N}$ and $\mathcal{M}$ by setting*

$$\big(\varphi(g)\big)(A) := \beta\big(g\big(\beta^{-1}(A)\big)\big),$$

where A is any point of the Poincaré disk and $g \in \mathcal{N}$.

10.1.5. Construction of perpendiculars in the Cayley–Klein model. The fact that we have a concrete isomorphism between the two disk models can be used to construct the "strange looking perpendiculars" (look at Figure 9.4 again) in the Cayley–Klein model. To do that, we use the bijection β from 10.1.2 to pass to the Poincaré

disk model, where we know how to construct perpendiculars (see Theorem 7.4.1) and, having performed that construction, we return to the Cayley–Klein model via β^{-1}, obtaining the required perpendiculars.

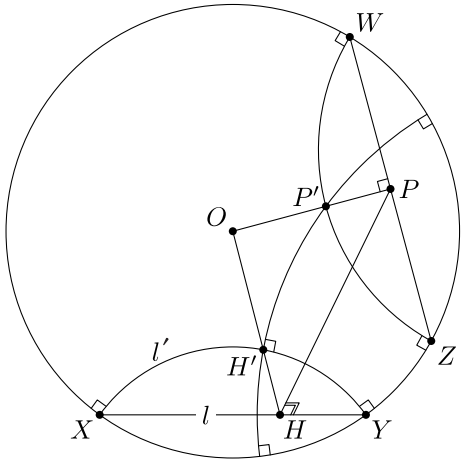


Figure 10.4. Constructing perpendiculars in the Cayley–Klein model.

In more detail, the construction is as follows (Figure 10.4). We are given a line $l = XY$ and a point P in the Cayley–Klein model $\mathbb{H}^2$. First we construct the chord WZ containing P and perpendicular to the radius OP. Next, we construct the two arcs of circles perpendicular to the absolute and passing through the points X, Y and W, Z and denote by P' the intersection point of the arc subtending WZ with the radius OP. Note that the two arcs are the images of the Cayley–Klein lines XY and WZ under the bijection β (see 10.1.2) and are therefore lines in the Poincaré disk model.

From the point P', we draw the arc orthogonal to the absolute and orthogonal to the arc l' subtending XY (see 7.4.1) and denote by H' the intersection point of these two arcs. Note that H' is the foot of the perpendicular drawn from P' to l' in the sense of the Poincaré disk model. Now if we construct the ray OH', its intersection point H with the line l is the foot of the required perpendicular drawn from P to l, because the map β^{-1} transforms the Poincaré perpendicular $P'H'$ to the Cayley–Klein perpendicular PH.

10.2. Isomorphism between the two Poincaré models

In this section, we show that the Poincaré disk model (Chapter 7) is isomorphic to the half-plane model studied in Chapter 8. To do that, we will need the linear-fractional transformation Ω defined (in Example 8.1) by the formula

$$\Omega : z \mapsto i \cdot \frac{1+z}{1-z};$$

Ω maps the unit disk $\mathbb{D}^2 := \{z \in \mathbb{C} : |z|^2 \leq 1\}$ to the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The transformation Ω, together with the compatibility (equivariance) condition determines the isomorphism between the two geometries. More precisely, we have the following result.

Theorem 10.2.1. *The map Ω from Example* 8.1 *defines an isomorphism of the geometry* $(\mathbb{H}^2 : \mathcal{M})$ (*the Poincaré disk model from Chapter* 7) *and the geometry* $(\mathbb{C}_+ : \mathbb{R}\text{Möb})$ (*the Poincaré half-plane model*) *if we define the corresponding isomorphism* (*which we denote by* Δ) *of the groups* $\mathbb{R}$Möb *and* $\mathcal{M}$ *by setting*

$$M \ni g \mapsto \Omega \circ g \circ \Omega^{-1} \in \mathbb{R} \text{ Möb}.$$

Proof. The map Ω is one-to-one because it has the obvious inverse given by the rule $w \mapsto (i-w)/(i+w)$. The isomorphism Δ is compatible with the group actions by definition. □

Now let us define the *Lobachevsky distance* λ between two points A, B of the open disk $\mathbb{H}^2$ (in the framework of the Poincaré disk model) by setting

$$\boxed{\lambda(A,B) := \frac{1}{2}|\log(\langle A, B, X, Y\rangle)|},$$

where X and Y are the intersection points of the line (AB) with the absolute and log stands for the natural logarithm.

Now Theorem 8.1.3, Lemma 8.1.5, and Theorem 10.2.1 immediately imply the following result:

Corollary 10.2.2. *The group of isometric transformations of the disk with respect to the distance λ coincides with the group $\mathcal{M}$ generated by all reflections in the "straight lines" of the disk model.*

Since isomorphism of geometries is a transitive relation, we have the following.

Corollary 10.2.3. *The three models of hyperbolic geometry, namely the Poincaré disk and half-plane models and the Cayley–Klein model, are isomorphic as geometries in the sense of Klein.*

10.3. Hyperbolic functions

The complex exponent e^z, $z \in \mathbb{C}$, is related to the ordinary trigonometric functions by the beautiful *Euler formula*:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi,$$

whose proof is obvious if we consider the unit circle centered at the origin of the plane $\mathbb{C}$. The real exponent e^x, $x \in \mathbb{R}$, is related to the "trigonometric functions" of hyperbolic geometry, known as the *hyperbolic functions* $\sinh, \cosh, \tanh, \coth$ (*hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent*, respectively) and defined by the formulas:

$$\sinh x := \frac{e^x - e^{-x}}{2}, \qquad \cosh x := \frac{e^x + e^{-x}}{2},$$

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x := \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

These functions satisfy formulas similar to the main formulas for ordinary trigonometric functions. Here are some examples:

$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh x \coth x = 1,$$

$$s\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\sinh 2x = 2\sinh x \cosh x, \quad \cosh 2x = \sinh^2 x + \cosh^2 x,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

The proofs are obtained by plugging the definitions into the formulas and performing simple calculations.

10.4. Trigonometry on the hyperbolic plane

Because of Corollary 10.2.3, the elementary trigonometric formulas for hyperbolic triangles are exactly the same for the half-plane and the disk model. Their proof is quite straightforward (perhaps a little simpler in the case of the half-plane) and are relegated to the problems. We state them in the form of theorems. Below ABC is a triangle, α, β, γ are the angles A, B, C, respectively, and a, b, c the sides opposite to A, B, C, respectively.

Theorem 10.4.1 (Hyperbolic sine theorem)**.**

$$\boxed{\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}}.$$

Theorem 10.4.2 (Hyperbolic cosine theorem)**.**

$$\boxed{\cosh a = \cosh b \cosh c - \sinh c \sinh b \cos\alpha}.$$

10.5. Angle of parallelism and Schweikart constant

10.5.1. Let (AB) be a line in hyperbolic geometry (we can use either one of the two models here) and C a point not on (AB); let X and Y be the intersection points of the line (AB) with the absolute, so that the rays $[CX)$ and $[CY)$ are the parallels to (AB) passing through C; let $[CH]$, $H \in (AB)$, be the perpendicular dropped from C to (AB); let $d := \lambda(C, H)$ be the Lobachevsky distance between C and H; finally, let α be the measure of the angle XCH (or, which is the same, of YCH). (See Figure 10.5, left.)

Then it is not difficult to prove that α depends only on d (see Problem 10.11); α is called the *angle of parallelism*.

Theorem 10.5.2. *The angle of parallelism α is given by the formula:*

$$\boxed{\tanh d \;=\; \cos\alpha}.$$

For the proof, see Problem 10.9.

This formula shows, in particular, that when d is very small, the angle of parallelism is close to $\pi/2$, while for large values of d, α becomes very small.

10.5.3. Now let O be the center of the disk model and let $[OA)$ and $[OB)$ be perpendicular rays issuing from O; let X and Y be the intersection points of the rays $[OA)$ and $[OB)$ with the absolute; let (CD) be the line intersecting the absolute at X and Y; finally, let $[OH]$, $H \in (CD)$, be the perpendicular from O to (CD); let $\sigma := \lambda(O, H)$ be the hyperbolic distance between O and H (Figure 10.5, right).

The number σ is called the *Schweikart constant*; it is an absolute constant of the hyperbolic plane. If we think of hyperbolic geometry as a model of physical reality, then we must conclude that there is an absolute unit of length in our universe (no such unit appears in the Euclidean model of space).

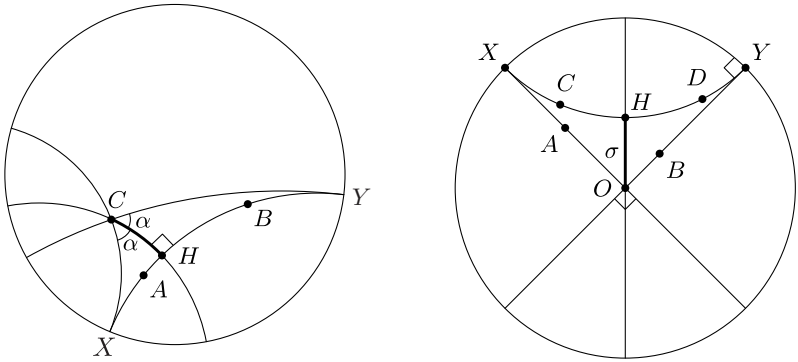


Figure 10.5. The angle of parallelism and the Schweikart constant.

10.5.4. Another absolute constant of hyperbolic geometry comes from the measure of a standard area, namely that of a special infinite "triangle". To construct this triangle, consider three rays issuing from the center (actually, any other point will do) of the disk model and forming angles of $2\pi/3$. Denote by X, Y, Z their intersection points with the absolute, and consider the lines XY, YZ, ZX. They form an "infinite equilateral triangle" with all three angles equal to zero. Then its area can be computed by the formula for the area of a triangle in hyperbolic geometry

$$S = \pi - \alpha - \beta - \gamma \quad \Longrightarrow \quad S = \pi$$

(see Chapter 8 and Problem 8.6).

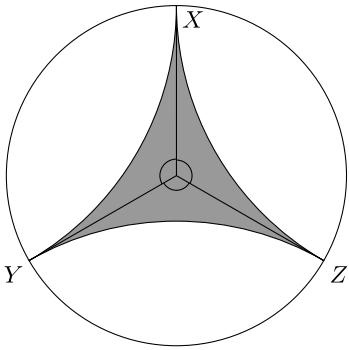


Figure 10.6. Infinite triangle of area π.

The above argument was not very rigorous, since the formula used is applicable only to finite triangles, but it can be made rigorous by approximating triangle XYZ by finite triangles and passing to the limit.

Thus we have obtained a third absolute constant, namely π, the area of the figure bounded by three lines joining three points of the absolute.

10.5.5. Remark. We noted above (see Section 9.4) that the formula for adding vectors on the hyperbolic line is very similar to Einstein's formula for adding the velocities of inertial frames. In this section, we have obtained three absolute constants – this is another trait of hyperbolic geometry that is similar to the properties of Einstein's theory of the physical world, in which absolute constants (e.g. the speed of light) appear. In this connection, one should not be misled by the word "relativity": Einstein's theory doesn't say that "everything is relative"; on the contrary, it supplies us with physically meaningful absolute constants, something that a Euclidean model of the universe cannot do. On the other hand, a physical model entirely based on hyperbolic space geometry and an independent "time axis" is not viable either: our universe is more complicated than that, time and space are not independent, according to Einstein, they "mingle together" in a certain sense.

10.6. Problems

10.1. Prove that stereographic projection is conformal (i.e., it preserves the measure of angles).

10.2. Prove that the map β constructed in 10.1.2 is bijective and show that any chord of $\mathbb{H}^2$ (i.e., any line in the Cayley–Klein model) is taken by β to the arc of the circle passing through X and Y and orthogonal to the absolute (i.e., to a line in the Poincaré disk model).

10.3. Prove the main relations between the hyperbolic functions indicated in Section 10.3.

10.4. Prove the hyperbolic sine theorem.

10.5. Prove the hyperbolic cosine theorem.

10.6. Prove that two triangles with equal sides are congruent in hyperbolic geometry.

10.7. Prove that in hyperbolic geometry two triangles having an equal angle and equal sides forming this angle are congruent.

10.8. Show that homothety is not conformal in hyperbolic geometry.

10.9. (a) Prove the formula for the angle of parallelism α for a point A and a line l:

$$\tanh d = \cos\alpha,$$

where d is the distance from A to l (thereby showing that the angle of parallelism depends only on the distance from the point to the line).

(b) Prove that the previous formula is equivalent to the following one (obtained independently by Bolyai and Lobachevsky):

$$\tan\frac{\alpha}{2} = e^{-d}.$$

10.10. Prove that in a triangle with right angle γ the sides a, b, c and their opposite angles $\alpha, \beta, \gamma = \pi/2$ satisfy the following relations:

$$\sinh a = \sinh c \sin\alpha, \quad \tanh b = \tanh c \cos\alpha,$$

$$\cot\alpha \cot\beta = \cosh c, \ \cos\alpha = \cosh a \sin\beta.$$

What do these relations tend to as a, b, c become very small?

10.11. Prove that the sides a, b, c and opposite angles α, β, γ of any triangle on the hyperbolic plane satisfy the following relations:

(a) $\cosh a \sin \beta = \cosh b \sin \alpha \cos \beta + \cos \alpha \sin \gamma$,

(b) $\cosh a = \dfrac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}$.

10.12. Prove that if the respective angles of two triangles are equal, then the triangles are congruent.

10.13. Prove that all the points of the (Euclidean) straight line $y = kx$ that lie in the upper half-plane $y > 0$ are equidistant from the (hyperbolic) straight line Oy.

10.14. (a) Prove that any hyperbolic circle contained in any one of the Poincaré models of hyperbolic geometry is actually a Euclidean circle.

(b) For the Poincaré upper half-plane model, find the Euclidean center and radius of the hyperbolic circle of radius r centered at the point (a, b).

(c) For the Poincaré model in the unit disk D, find the relationship between the radii of the Euclidean and the hyperbolic circles centered at the center of D.

10.15. Prove the triangle inequality for the distance in the Poincaré half-plane model.

10.16. Prove that the three (a) bisectors, (b) medians, (c) altitudes of any hyperbolic triangle intersect at one point.

10.17. (*The hyperbolic Menelaus Theorem*) The line l intersects the lines BC, CA, AB (containing the sides) of triangle ABC at the points A_1, B_1, C_1 respectively; then

$$\frac{\sinh AC_1}{\sinh C_1B} \frac{\sinh BA_1}{\sinh A_1C} \frac{\sinh CB_1}{\sinh B_1A} = 1.$$

10.18. (*The hyperbolic Ceva Theorem.*) The points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of triangle ABC. Prove that the

segments AA_1, BB_1, CC_1 intersect at one point if and only if one of the following two equivalent conditions holds:

$$\frac{\sin ACC_1}{\sin C_1CB} \cdot \frac{\sin BAA_1}{\sin A_1AC} \cdot \frac{\sin CBB_1}{\sin B_1BA} = 1,$$

$$\frac{\sinh AC_1}{\sinh C_1B} \cdot \frac{\sinh BA_1}{\sinh A_1C} \cdot \frac{\sinh CB_1}{\sinh B_1A} = 1.$$

Chapter 11

History of Non-Euclidean Geometry

In this chapter, we will retrace the history of the creation of non-Euclidean geometry by Gauss, Lobachevsky, and Bolyai (and their predecessors and followers) and discuss the traditional axiomatic approach to the foundations of geometry. The story begins with Euclid's *Elements*, the brilliant first attempt to construct mathematics as a deductive science (see [8] and Appendix A).

11.1. Euclid's Fifth Postulate

The Ancient Greeks realized that, in a deductive science, in order to deduce (prove) facts from other facts by logical reasoning, it is necessary to start from some facts which are not proved. Euclid called these facts *postulates* (we call them *axioms*) and explicitly formulated five of them. He also used several other axioms implicitly (without formulating them). Apparently, Euclid (and other Greek mathematicians) thought that the postulates should be self-evident (simple and so obvious that no doubt about their truth could arise).

Euclid's last axiom, the *Fifth Postulate*, however, is not simple and not obvious. Its modern equivalent can be stated as follows.

(V+) *For any straight line and any point not on this line there is a unique parallel to this line passing through the given point.*

Here by a *parallel* to a given line one means a straight line that has no common points with the given line. In Euclid's formulation, the statement was more complicated and less obvious.

(V) *If a straight line falling on two straight lines makes the sum of the interior angles on one side less than two right angles, then the two straight lines, if extended indefinitely, meet on that side on which the angles with sums less than two right angles exist.*

Presumably, Greek mathematicians (perhaps Euclid himself) tried to deduce the Fifth Postulate from the other axioms. In any case, in Euclid's *Elements*, the application of the Fifth Postulate is postponed as much as possible: it occurs for the first time in the proof of Proposition 27 of Book 1 (there are 48 propositions, i.e., theorems in our terminology, in that book). The interested reader may want to look at the postulates and theorems in Book 1 of Euclid's *Elements*: they appear in Appendix A of the present book.

After Euclid, for more than two thousand years, many scientists tried to prove the Fifth Postulate, and many "succeeded", usually by proving statements equivalent to (V) by means of arguments based on additional axioms which were not explicitly formulated.

11.2. Statements equivalent to the Fifth Postulate

We have already mentioned one such statement, namely (V+). Here are some more (in square brackets [ ], we indicate the mathematician who used this approach to "prove" the Fifth Postulate).

(1) *The sum of the three angles of any triangle is equal to* π (*to two right angles,* in Euclid's terminology). [This statement appears in Euclid's *Elements* as Proposition 32, and was proved by using the Fifth Postulate; Legendre gave a "proof" in 1805 without the Fifth Postulate.]

(2) *A line intersecting one of two parallel lines intersects the other* [Proclus, 5th century].

(3) *Similar but not congruent triangles exist* [John Wallis, 1663].

(4) *The fourth angle of a quadrilateral with three right angles is also a right angle* [Nasiraddin, 13th century; Saccheri, 1773; Lambert, 1766] . Such a quadrilateral was later called a *Saccheri quadrilateral*.

Trying to prove the Fifth Postulate, most mathematicians (including those mentioned above) argued by contradiction. As a rule, they considered two cases, assuming that the sum of angles of a triangle is (a) more than π or (b) less than π (equivalently, that the fourth angle of the Saccheri quadrilateral is more (less) than $\pi/2$, or that there are no parallels, respectively, more than one parallel, through a given point to a given line). In the first case, it is possible to correctly obtain a contradiction using the Euclidean axioms. In the second case, a contradiction does not follow, but the desire to prove the Fifth Postulate was so strong that the mathematicians working on the problem usually produced what they claimed to be a proof, but which was actually flawed.

11.3. Gauss

Carl Friedrich Gauss (1777–1855) first began working on the Fifth Postulate in 1796, at the age of nineteen, and argued by contradiction, like his predecessors, but went much further in developing the theory in case (b). It is not clear when he came to the conclusion that no contradiction would arise. In a famous letter (1824) to his friend F.A. Taurinus, he explained that in the case $\alpha + \beta + \gamma < \pi$ one obtains a "thoroughly consistent curious geometry", which he called "non-Euclidean". He concluded his letter by asking Taurinus not to tell anyone about his "private communication", which he was thinking of publishing at "some future time".

Later, in 1832, he learned from his friend Farkas Bolyai that the latter's son, Janos, had arrived at the same conclusions. Later still, in 1841, he found out that Lobachevsky had done the same. Gauss even learned Russian (to read Lobachevsky's early work?), but never directly communicated with either Janos Bolyai or Lobachevsky about these questions.

Carl Friedrich Gauss

The most amazing thing, however, is that Gauss, when he was not thinking about number theory or the Fifth Postulate, had constructed the differential geometry of surfaces, including surfaces of constant negative curvature, which are, in fact, a model (at least locally) of hyperbolic geometry. All these years, he had this model before his eyes, but never made the obvious connection with non-Euclidean geometry. He died without suspecting that a proof of the consistency of hyperbolic geometry was at his finger tips!

11.4. Lobachevsky

Nikolai Ivanovich Lobachevsky (1793–1856), like everybody else, tried to prove the Fifth Postulate by contradiction. As he progressed further in the case $\alpha + \beta + \gamma < \pi$, he became convinced that the theory

was consistent. In an unpublished textbook, written in 1823, he mentions that all attempts to prove the Fifth Postulate were erroneous. In 1826, Lobachevsky published a memoir about a new geometry (which he called imaginary) in the Kazan Bulletin, but this publication (written in Russian) went unnoticed abroad. Trying to gain recognition, he published his work in German (*Geometrische Untersuchugen*, 1840) and in French (*Pangéométrie*, 1855), but without success (for an English translation of his work, see [11]).

Nikolay Ivanovich Lobachevsky

N.I. Lobachevsky was not only the President (Rector, in the Russian terminology) of Kazan University, but also its Head Librarian. The Kazan library received many scientific periodicals, including the

most famous mathematical journal of the time, *Crelle's Journal*. Library cards (which have come down to us) show that Lobachevsky read every issue of Crelle's Journal that reached Kazan, except two successive issues in the 1830s. These two issues contained two papers by Mindling, in which the latter obtained, on surfaces of constant negative curvature, trigonometric formulas identical to the trigonometric formulas previously obtained by Lobachevsky on the hyperbolic plane. Had Lobachevsky seen one of these papers, it is very likely that he would have observed that they could be used to obtain a proof of the consistency of hyperbolic geometry!

11.5. Bolyai

Janos Bolyai (1802–1860) was the son of a mathematician, Farkas Bolyai, who had "proved" the Fifth Postulate (his friend Gauss had pointed out his error). Janos first followed in his father's footsteps by trying to prove the Fifth Postulate by contradiction, but soon realized that he was obtaining a consistent geometry. In 1823 he wrote to his father: "Out of nothing I have created a strange new universe." But it was only in 1832 (three years after Lobachevsky) that his investigations were published in an Appendix to his father's book *Tentamen* (both were written in Latin; for the German translation, see [15], the English translation of the Appendix appeared in [17] and in [14], p. 375).

Farkas sent the book to Gauss, asking to comment on the Appendix. Instead of praising and encouraging Janos, Gauss wrote that this would be "praising myself", since he had discovered the same things thirty years before, and the Appendix "spared him the effort" of writing up his discovery. Discouraged, Janos Bolyai stopped working for several years, but then started working on a book that would contain a detailed exposition of his results.

When Gauss had learned about Lobachevsky's results, he "kindly" communicated this fact to Janos Bolyai via the latter's father. For a while, Janos thought that Lobachevsky did not exist, that he was a creation of Gauss, who used "Lobachevsky" as a pen name to publish results stolen from J. Bolyai's Appendix! Fortunately, Janos Bolyai finally understood that this was not the case, but he never finished

Janos Bolyai

his book; in fact, he published nothing more. He died fairly young, unrecognized by his contemporaries...

11.6. Beltrami, Helmholtz, Lie, Cayley, Klein, Poincaré

The first proof of the consistency of hyperbolic geometry is due to Beltrami, who showed (1868) that its axioms and theorems hold (at least locally) on surfaces of constant negative curvature. Recent research in the history of the subject shows that Beltrami was also aware of what is commonly known as the Poincaré and Cayley–Klein models. The physicist Helmholtz was probably the first to understand how one can prove the consistency of hyperbolic geometry, but his arguments were regarded as insufficiently rigorous by mathematicians. Sophus

Lie improved the arguments of Helmhotz and was the first to stress the role of transformation groups in mathematics. Klein gave the definition of geometry that we introduced in Chapter 1, and, simultaneously with Cayley (but independently of him), gave the elementary global model of hyperbolic geometry described in Chapter 9; he also coined the terms *hyperbolic, parabolic, elliptic* for the three geometries. Poincaré constructed the two models of hyperbolic geometry that we discussed in Chapters 7 and 8.

11.7. Hilbert

David Hilbert made the first successful attempt to give an axiomatic exposition of Euclidean (space) geometry, rigorous in the modern sense of the word. It consists of 21 axioms, three undefined concepts (*point, line, plane*), and several undefined relations. Hilbert's axioms for plane geometry are presented and discussed in Appendix B of the present book.

Hilbert's axiomatic approach is rarely used in teaching geometry in our time, because Euclidean geometry can be introduced in a much simpler way: it can easily be constructed as a branch of linear algebra over the real numbers (based on the fact that the straight line is "isomorphic" to the real numbers $\mathbb{R}$). This fact can be deduced from Hilbert's axioms by using the axiomatic definition of the real numbers and checking that these algebraic axioms are satisfied by the points of any line, provided the product and sum operation are appropriately defined on it.

Chapter 12

Projective Geometry

In this chapter, we introduce the main ideas of projective geometry for the particular case of $\mathbb{R}P^2$, the projective plane, and we only take a brief look at the projective space $\mathbb{R}P^3$. The general theory of d-dimensional projective spaces ($\mathbb{R}P^d$, $d \geq 1$) is traditionally studied in linear algebra courses by means of the so-called homogeneous coordinate model, but we do not go beyond the dimension $d = 3$. We use a more geometric approach, which may seem strange at first, because in our model "points" of $\mathbb{R}P^2$ will be lines in Euclidean space $\mathbb{R}^3$, but ultimately we will also appeal to the homogeneous coordinate model.

12.1. The projective plane as a geometry

12.1.1. Main definition. The *projective plane* $\mathbb{R}P^2$ is defined as the geometry $(\mathbb{R}P^2 : \mathrm{Proj}(2))$, whose elements (called *projective points*) are straight lines in $\mathbb{R}^3$ passing through the origin O and whose transformation group Proj(2) is defined as follows. We start with the general linear group GL(3) and identify any two linear transformations of $\mathbb{R}^3$ whose matrices can be obtained from each other by multiplication by nonzero constants; the composition of matrices is well defined on such equivalence classes of transformations, and Proj(2) is defined as the group whose elements are these classes and the group operation is composition (i.e., multiplication of matrices).

12.1.2. Points and lines. The elements of $\mathbb{R}P^2$ (projective points) are Euclidean lines; nevertheless, we will often simply call them *points* (of our geometry). The *straight lines* (of our geometry) are defined as the (Euclidean) planes passing through the origin. These definitions immediately imply the following two assertions.

I. *One and only one "line" passes through any two distinct "points".*

II. *Any two distinct "lines" intersect in one and only one "point".*

Thus there are no parallel lines in our geometry, just as in spherical geometry. But we will see that the two geometries are very different; in particular, there is no natural metric in projective geometry (and hence no measure of angles, no perpendiculars, no areas, and so on). Unlike spherical geometry, in which "straight lines" intersect in two points, in projective geometry lines intersect in one point, not two.

12.1.3. Intuitive description. You can imagine the projective plane as a Euclidean plane to which a "line at infinity" Λ_∞ has been added. When you move along a Euclidean line L to infinity in some direction, you intersect the line at infinity at some point $P = L \cap \Lambda_\infty$; if you move along L in the opposite direction, you will reach Λ_∞ and intersect it *at the same point* P. Parallels (in the Euclidean sense) intersect on the infinite line. Thus lines in $\mathbb{R}P^2$ are some kind of cycles (like "infinite circles"). The line at infinity, however, should not be regarded as a "special" line, because most projective transformations transform it into an "ordinary" line. The informal description of $\mathbb{R}P^2$ given here will be made rigorous in Subsection 12.2.4.

12.2. Homogeneous coordinates

12.2.1. Returning to our geometry $(\mathbb{R}P^2 : \mathrm{Proj}(2))$, let us introduce coordinates for our points. Each point L (i.e., each Euclidean line passing through the origin) is uniquely determined by its *direction vector*, i.e., by three coordinates (x_1, x_2, x_3), in the standard basis of $\mathbb{R}^3$, namely in the basis

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

Conversely, however, points *do not* uniquely determine the coordinates: if λ is a nonzero real number, then $(\lambda x_1, \lambda x_2, \lambda x_3)$ determines the same point as (x_1, x_2, x_3). In this situation, we call the two sets of coordinates *equivalent*, denote the corresponding equivalence class by $(x_1 : x_2 : x_3)$, and refer to $\chi(L) = (x_1 : x_2 : x_3)$ as the *homogeneous coordinates* of the point L.

12.2.2. Homogeneous coordinates make the computation of the action of elements $g \in \mathrm{Proj}(2)$ on points $L \in \mathbb{R}P^2$ very easy: the transformation g is given by a 3×3 matrix $A_g \in \mathrm{GL}(3)$ (defined up to a constant), and

$$g(L) = A_g((x_1 : x_2 : x_3)) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The geometric meaning of the transformation with matrix A_g is that its column vectors are the images of the standard basis vectors under that transformation, but since A_g is defined up to a nonzero scalar, these images are also defined up to a nonzero scalar multiple.

12.2.3. Projective spaces of higher dimensions. In linear algebra courses, the *projective space* $\mathbb{R}P^d$, for any value of d, is defined in a similar way: its elements are homogeneous coordinates

$$(x_0 : x_1 : \cdots : x_d),$$

i.e., equivalence classes of $(d+1)$-tuples $(x_0 : x_1 : \cdots : x_d)$ of real numbers (not all equal to zero) up to multiplication by a nonzero constant. The group $\mathrm{Proj}(d+1)$ acts on each element by multiplication by $(d+1) \times (d+1)$ matrices corresponding to linear operators in $\mathbb{R}^{d+1}$ (defined up to a constant). We will not study higher-dimensional projective spaces $\mathbb{R}P^d$, $d > 3$, in this course. A detailed account can be found in most linear algebra courses. However, we will look at projective space $\mathbb{R}P^3$ briefly in Section 12.8 below.

12.2.4. Now let us describe a rigorous model of $\mathbb{R}P^2$ that will explain why $\mathbb{R}P^2$ is called the projective *plane*. In $\mathbb{R}^3$ consider the plane Π given by the equation $x_3 = 1$. Points of this plane have coordinates

of the form $(x_1, x_2, 1)$. To the plane Π add the *line at infinity* Λ_∞ whose *points* are equivalence classes of Euclidean points $(x_1, x_2, 0)$ up to multiplication by a nonzero constant (notation $(x_1 : x_2 : 0)$). The set $\Pi \cup \Lambda_\infty$ is the set of points of the projective plane.

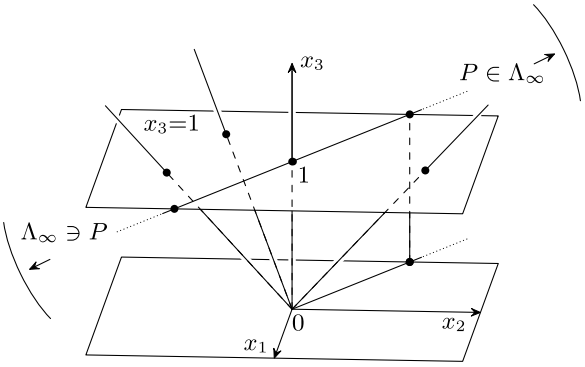


Figure 12.1. The projective plane.

Note that the "points at infinity" $(x_1 : x_2 : 0) \in \Lambda_\infty$ determine Euclidean straight lines in the plane $x_3 = 0$. Intuitively, you should think of these lines as "pointing to infinity" in a certain direction, so that the set Λ_∞ "surrounds" the plane Π. More precisely, these lines are not rays, they are ordinary "two-sided" lines, and so they point to infinity in two opposite directions, but they intersect the projective line Λ_∞ at only one point (you should think of this point as being the identification of two diametrically opposite points at infinity).

The *lines* in this model of $\mathbb{R}P^2$ are the ordinary (Euclidean) lines in Π plus the "line" Λ_∞. There is an obvious bijection between the points and lines of $\mathbb{R}P^2$ (as defined in the previous section) and those in the model $\Pi \cup \Lambda_\infty$; in particular, the line Λ_∞ corresponds to the (Euclidean) plane $x_3 = 0$. Using this bijection, it is easy to define the action of Proj(2) in this model.

12.3. Projective transformations

12.3.1. One may want to ask: Why is our geometry called "projective", when it is defined by a group of *linear* operators in $\mathbb{R}^3$? Let us

try to answer this question. Let Π_1 and Π_2 be two planes in $\mathbb{R}^3$ and let $P \in \mathbb{R}^3$ be a point. The *projection* of Π_1 to Π_2 from P is the map π that to each point $A \in \Pi_1$ assigns the point $A' \in \Pi_2$ at which the line PA intersects Π_2. This assignment is not necessarily bijective: π will be undefined at some points X (if PX is parallel to Π_2) and not onto (some points of Π_2 will not be covered); see Figure 12.2.

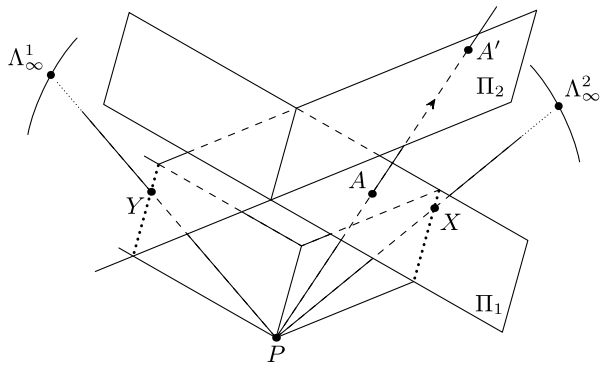


Figure 12.2. Projective transformations of planes.

However, if we supply Π_1 and Π_2 with lines at infinity Λ^1_∞ and Λ^2_∞, and appropriately define the projection, then we obtain a bijection between the projective planes $\Pi_1 \cup \Lambda^1_\infty$ and $\Pi_2 \cup \Lambda^1_\infty$. The details are left to the reader.

12.3.2. A set of points $A_1, \ldots, A_n$, $n \geq 3$, of the projective plane $\mathbb{R}P^2$ (interpreted as the model described in Subsection 12.2.4) are said to be *in general position* if for any three of them, A_k, A_l, A_m, the vectors $\overrightarrow{OA}_k, \overrightarrow{OA}_l, \overrightarrow{OA}_m$ constitute a basis of $\mathbb{R}^3$. If one of the points, say A_i, lies on the line at infinity, the vector $\overrightarrow{OA}_1$ is well defined, in coordinates it has the form $(a : b : 0)$. If three points or more from our collection lie on the infinite line, then, of course, the collection will not be in general position.

Another way of defining a collection of points in general position is to say that no three of them lie on the same line.

Theorem 12.3.3. *There exists one and only one projective transformation that takes four points $A, B, C, D \in \mathbb{R}P^2$ in general position to four other points $A', B', C', D' \in \mathbb{R}P^2$ in general position.*

Proof. In accordance with our model of the projective plane, we can think of the points A, B, C and A', B', C' as lying in the plane $x_3 = 1$. By assumption, the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ constitute a basis of $\mathbb{R}^3$. Let (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) be the coordinates of the vectors $\overrightarrow{OA'}, \overrightarrow{OB'}, \overrightarrow{OC'}$ in that basis. Then the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

can be regarded as a linear transformation of $\mathbb{R}^3$ taking A, B, C to A', B', C'. Now let us multiply the columns of this matrix by scalar constants, obtaining the matrix

$$A_g = \begin{pmatrix} \lambda a_1 & \mu b_1 & \nu c_1 \\ \lambda a_2 & \mu b_2 & \nu c_2 \\ \lambda a_3 & \mu b_3 & \nu c_3 \end{pmatrix}$$

which we now regard as defining an element g of Proj(2). Clearly, A_g takes the points $A, B, C \in \mathbb{R}P^2$ to the points $A', B', C' \in \mathbb{R}P^2$, although the same matrix regarded as acting in $\mathbb{R}^3$ *does not* take $A, B, C \in \mathbb{R}^3$ to $A', B', C' \in \mathbb{R}^3$ (when not all three of the scalars λ, μ, ν are equal to 1).

Now let us denote by (d_1, d_2, d_3) the coordinates of the point D in the basis $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and by (d'_1, d'_2, d'_3) the coordinates of the point D' in the same basis. We claim that it is possible to choose the scalar parameters λ, μ, ν so that A_g will take $D \in \mathbb{R}P^2$ to $D' \in \mathbb{R}P^2$.

Indeed, this will be the case if the matrix A_g applied to the vector (d_1, d_2, d_3) will give the vector (d'_1, d'_2, d'_3), or, which is the same thing, the system of equations,

$$\begin{cases} a_1 d_1 \lambda + b_1 d_2 \mu + c_1 d_3 \nu = d'_1, \\ a_2 d_1 \lambda + b_2 d_2 \mu + c_2 d_3 \nu = d'_2, \\ a_3 d_1 \lambda + b_3 d_2 \mu + c_3 d_3 \nu = d'_3, \end{cases}$$

in the unknowns λ, μ, ν will have a solution. But the determinant Δ of this system can be expressed as $\Delta = d_1 d_2 d_3 \det(M)$ and so, it is nonzero. Hence our system of equations has a nonzero solution in λ, μ, ν. Thus we have shown that $A_g(D) = D'$ (if we choose for the values of λ, μ, ν the solution of our system) and proved the existence of the required projective transformation.

Its uniqueness follows by working out the construction of A_g in reverse order, which will bring us back to the same matrix (up to multiplication by a scalar). □

12.4. Cross ratio of collinear points

12.4.1. Main definitions. We mentioned above that there is no natural metric on the projective plane, and no affine structure (the ratio of the two segments determined by *three* collinear points of $\mathbb{R}P^2$ is not well defined). Nevertheless, the affine structure in $\mathbb{R}^3$ allows us to define the cross ratio of any *four* ordered collinear points of $\mathbb{R}P^2$.

The definition is the following. Let k, l, m, n be collinear points in $\mathbb{R}P^2$, i.e., four coplanar lines of $\mathbb{R}^3$ passing through the origin; suppose a line s cuts our four lines at the points A, B, C, D, respectively. Then the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are proportional, i.e., $\overrightarrow{AC} = \lambda \overrightarrow{BC}$; the real number λ (which may be negative) is denoted by $\langle A, B, C\rangle$; the number $\langle A, B, D\rangle$ is defined similarity. We now put

$$\langle A, B, C, D\rangle := \frac{\langle A, B, C\rangle}{\langle A, B, D\rangle};$$

the number thus obtained is called the *cross ratio* of four collinear points A, B, C, D. It is not difficult to show that it is well defined, i.e., does not depend on the choice of the secant line s. Now if one of the points, say B, lies on the infinite line Λ_∞, then we put $\langle A, B, C, D\rangle := \langle C, D, A\rangle$ (similarly for the other cases).

12.4.2. Coordinate expressions. The cross ratio is easy to compute in coordinates. To this end, we return to the model

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1\} \subset \mathbb{R}P^2 = \Pi \cup \Lambda_\infty$$

and suppose that the collinear points A, B, C, D have the coordinates:

$$(x_A, y_A, 1),\ (x_B, y_B, 1),\ (x_C, y_C, 1),\ (x_D, y_D, 1).$$

Then, obviously,

$$\langle A,B,C\rangle = \frac{x_C - x_A}{x_C - x_B} = \frac{y_C - y_A}{y_C - y_B},\ \langle A,B,D\rangle = \frac{x_D - x_A}{x_D - x_B} = \frac{y_D - y_A}{y_D - y_B}$$

and therefore

$$\langle A,B,C,D\rangle = \frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - x_B} = \frac{y_C - y_A}{y_C - y_B} : \frac{y_D - y_A}{y_D - y_B}.$$

If one of the points, say B, is on the infinite line (at its intersection with the line containing the points A, C, D), then the cross ratio reduces to the ordinary ratio. What happens in this case may be described by saying that "the infinities cancel":

$$\frac{x_C - x_A}{x_C - \infty} : \frac{x_D - x_A}{x_D - \infty} = \frac{x_C - x_A}{x_D - x_A} = \langle C,D,A\rangle.$$

In the case when all four points A, B, C, D lie on the infinite line, their cross ratio is also a well-defined real number. Its calculation is the object of Problem 12.3.

Theorem 12.4.3. *The cross ratio of four collinear points is invariant under projective transformations.*

Proof. The proof is a problem in linear algebra; see Problem 12.4. □

12.5. Projective duality

12.5.1. Points and lines on the projective plane $(\mathbb{R}P^2 : \mathrm{Proj}(2))$ play, in a certain sense, symmetric roles. This will be easier to see if we introduce the notion of *incidence*: we will say that two lines a and b are *incident at the point* P if P is the intersection point of the lines a and b, and that the two points P and Q are *incident to the line* a if a passes through P and Q. Also, together with the standard term *collinear* (used for points all lying on one line) we will use the term *copunctal* for lines all passing through one and the same point.

Given an assertion of projective geometry formulated in this terminology, we can translate it into another statement, called *dual*, by replacing the word "line" by the word "point" (and "collinear" by "'copunctal") and vice versa. For example, statement I from Section 12.1 can be expressed as: "One and only one line is incident to

two distinct points"; its translation (i.e., the dual statement) will be "One and only one point is incident to two distinct lines", which is exactly the assertion II (see Section 12.1). Another example: "Any projective transformation takes collinear points to collinear points" translates to "Any projective transformation takes copunctal lines to copunctal lines".

What is remarkable is that this kind of translation always translates true statements to true statements. To prove this, we will define the *dual geometry* to the geometry of $\mathbb{R}P^2$: it is the geometry $(D\mathbb{R}P^2 : \mathrm{Proj}(2))$ whose *points* are planes of $\mathbb{R}^3$ passing through the origin under the action of the group of linear nonsingular transformations of $\mathbb{R}^3$. In $D\mathbb{R}P^2$, the intersection of two points (i.e., Euclidean planes) will be called the *line passing through the points* (it is actually a Euclidean line in Euclidean 3-space).

Theorem 12.5.2. *The two geometries* $(D\mathbb{R}P^2 : \mathrm{Proj}(2))$ *and* $(\mathbb{R}P^2 : \mathrm{Proj}(2))$ *are isomorphic: there is a bijection, called duality and denoted by* D*, between the sets of points of the two geometries compatible with an isomorphism of* $\mathrm{GL}(3)$ *onto itself.*

Proof. To each "point" Π of $D\mathbb{R}P^2$, i.e., to each plane of $\mathbb{R}^3$ given by the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, we assign the point of $\mathbb{R}P^2$ with homogeneous coordinates $(a_1 : a_2 : a_3)$ (which is of course the Euclidean line passing through the origin and perpendicular to the plane). If an element $g \in \mathrm{Proj}(2)$ takes the point $(a_1 : a_2 : a_3)$ to some point $(b_1 : b_2 : b_3)$, then the same element will take the plane Π to the plane given by $b_1x_1 + b_2x_2 + b_3x_3 = 0$. Thus the duality map $D : \mathbb{R}P^2 \to D\mathbb{R}P^2$ (which is obviously bijective) is compatible with the action of $\mathrm{Proj}(2)$, so that we have constructed the required isomorphism. □

Note that the duality correspondence is an *involution*, i.e., $D \circ D$ identically maps $\mathbb{R}P^2$ onto itself. Further, note that the isomorphism constructed above preserves incidence: if two points A, B of $\mathbb{R}P^2$ (i.e., two Euclidean lines passing through the origin O of $\mathbb{R}^3$) are incident to the line l (i.e., are contained in a Euclidean plane Π_l), then the two lines $D(A), D(B)$ in $D\mathbb{R}P^2$ intersect in the point (of $D\mathbb{R}P^2$) $D(l) = \Pi_l$. Thus we have the following statement.

12.5.3. Corollary: Duality Principle. *There is a bijection between the set of lines and the set of points of $\mathbb{R}P^2$ that preserves incidence and takes any theorem of projective geometry to a theorem of projective geometry.*

12.6. Conics in $\mathbb{R}P^2$

The nondegenerate conic sections (or *conics* for short) in the Euclidean plane are, as is well known, the ellipse, the hyperbola and the parabola. In $\mathbb{R}P^2$, these three curves are projectively equivalent, so *there exists only one nondegenerate conic in $\mathbb{R}P^2$ (up to projective equivalence)*.

A conic in $\mathbb{R}P^2$ can be defined as any set of points obtained from the curve C given by $(x_1)^2 + (x_2)^2 = 1$ (in the plane-with-line-at-infinity model described in Section 12.2, this curve is the Euclidean circle) by a projective transformation. Any projective transformation under which the image of C does not intersect the line at infinity Λ_∞ transforms C into an ellipse; a projective transformation that takes one point of C to Λ_∞ transforms C into a parabola, and a projective transformation that takes two points of C to Λ_∞ transforms C into a hyperbola.

12.7. The Desargues, Pappus, and Pascal theorems

We conclude our study of $\mathbb{R}P^2$ with three beautiful classical theorems. All three can be regarded as theorems about points and lines either in the projective plane or in the affine (in particular Euclidean) plane.

12.7.1. Desargues' Theorem. *Suppose that the lines joining the corresponding vertices of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at one point S. Then the intersection points P_1, P_2, P_3 of the lines A_2A_3 and B_2B_3, A_3A_1 and B_3B_1, A_1A_2 and B_1B_2, respectively, are collinear.*

Proof. We begin by passing from the plane to 3-space and prove the three-dimensional analog of theorem Desargues'. (The proof of the 3D theorem turns out to be unexpectedly simple, but the argument

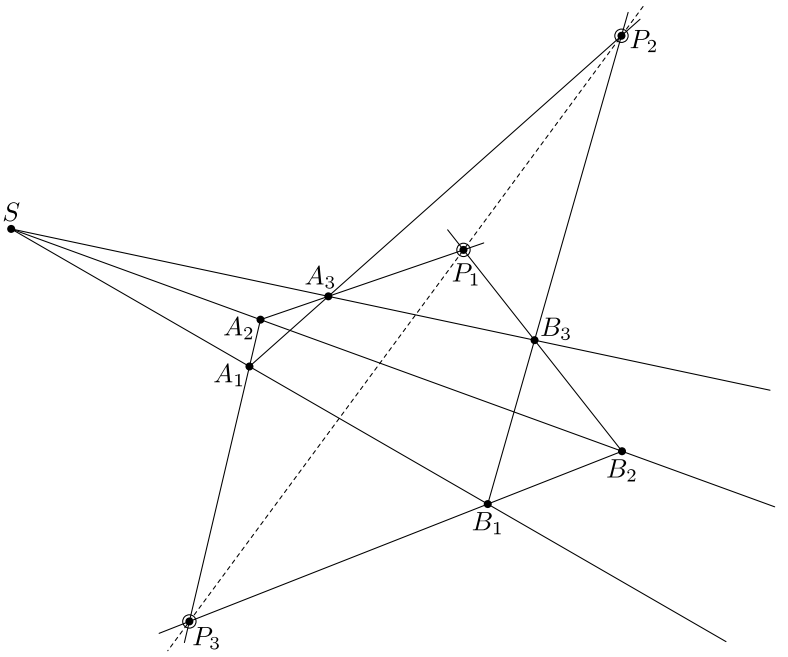


Figure 12.3. Desargues' theorem.

used in it doesn't work in the plane!) We then use the 3D theorem to prove Desargues' theorem in the plane by means of a continuous deformation of the spatial picture to the planar one.

Suppose we are given two triangles $A_1\widehat{A}_2A_3$ and $B_1\widehat{B}_2B_3$ in Euclidean space $\mathbb{R}^2$ such that the three lines A_1B_1, $\widehat{A}_2\widehat{B}_2$, A_3B_3 intersect at one point S. (The reader should think of the points A_1, B_1, A_3, B_3, S as being the same as in the planar version of the theorem, while the points A_2, B_2 have been "lifted out" of the plane.) Then the lines SB_1, $S\widehat{B}_2$, SB_3 define a trihedral angle in $\mathbb{R}^3$ (see Figure 12.4).

Consider the three pairs of lines $\widehat{A}_2A_3$ and $\widehat{B}_2B_3$, $\widehat{A}_2A_1$ and $\widehat{B}_2B_1$, A_1A_3 and B_1B_3. We claim that *each of these pairs has a common point (in space!) and these three points are collinear.*

Indeed, the (Euclidean) planes

$$\Pi_1 := (A_1\widehat{A}_2A_3) \quad \text{and} \quad \Pi_2 := (B_1\widehat{B}_2B_3)$$

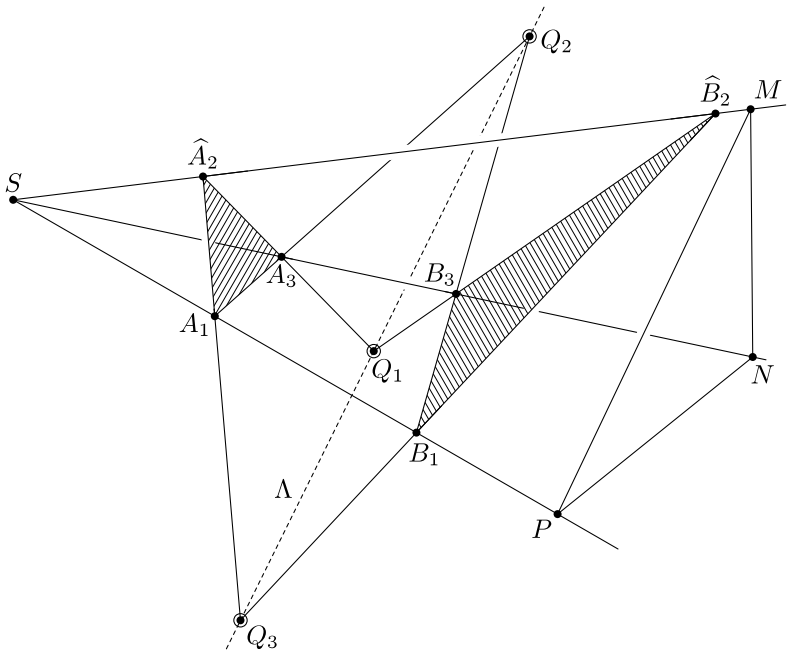


Figure 12.4. Desargues' theorem in space.

intersect in a line Λ. Obviously, the lines $\widehat{A}_2A_3$ and $\widehat{B}_2B_3$ intersect at a point (denoted Q_1) of Λ, and so do the lines $\widehat{A}_2A_1$ and $\widehat{B}_2B_1$ (the intersection point is denoted by Q_3) as well as the lines A_1A_3 and B_1B_3 (at Q_2). Since the points Q_1, Q_2, Q_3 all lie on Λ, they are collinear, as claimed.

Let us pass to the proof of the planar version of the theorem.

Consider the plane B_1SB_3 (which we think of as being "horizontal"), construct a plane perpendicular to it through the line SB_2, in that plane choose a point O "below" the horizontal plane, and choose points $\widehat{A}_2$ and $\widehat{B}_2$ so that S, $\widehat{A}_2$, $\widehat{B}_2$ are collinear by projecting the points A_2, B_2 from O (see Figure 12.5).

Using the 3D version of the theorem, we can now construct the three collinear points Q_1, Q_2, Q_3. Now rotate the line $S\widehat{B}_2$ about S downward in the vertical plane until it coincides with SB_2. Since the

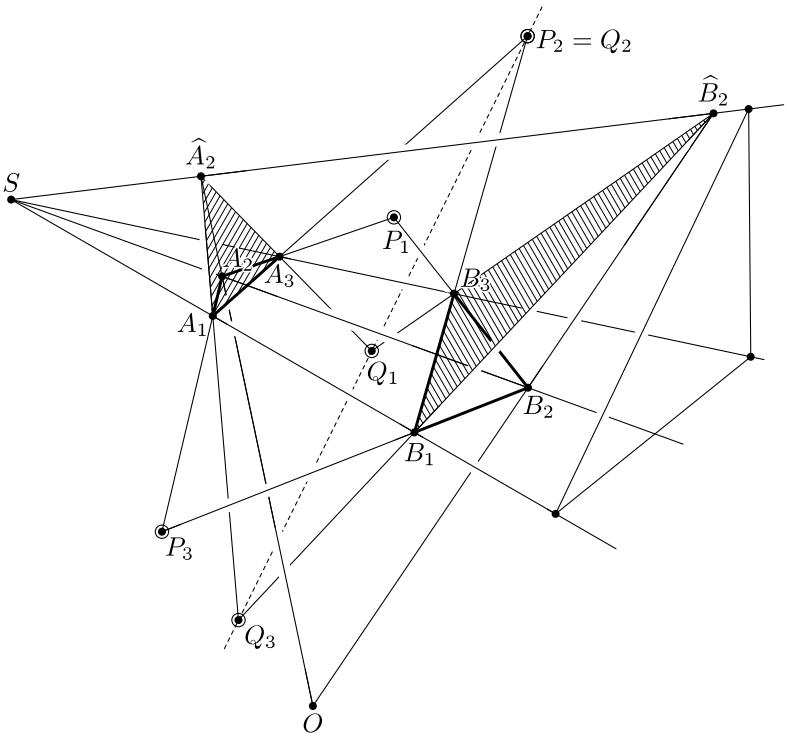


Figure 12.5. Proof of Desargues' theorem.

mobile points Q_1, Q_2, Q_3 will always be collinear and, when they reach the horizontal plane, they will coincide with the points P_1, P_2, P_3, it follows that these three points are collinear. This proves the theorem. □

12.7.2. Pappus' Theorem. *Suppose the points A_1, A_2, A_3 are collinear, and the points B_1, B_2, B_3 are collinear. Let P_1, P_2, P_3 be the intersection points of the lines A_2B_1 and A_1B_2, A_1B_3 and A_3B_1, A_2B_3 and A_3B_2, respectively. Then the points P_1, P_2, P_3 are collinear.*

Sketch of the proof. By Theorem 12.3.3, we can assume that $A_1A_2B_1B_2$ is a square. Using the coordinate system with basis $\overrightarrow{A_1A_2}$,

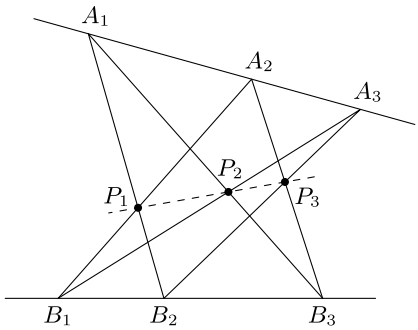


Figure 12.6. Pappus' theorem.

$\overrightarrow{A_1B_1}$, it is an easy exercise to prove that the points P_1, P_2, P_3 are collinear. □

12.7.3. Pascal's Theorem. *Suppose the points A, B, C, D, E, F lie on a conic. Let P_1, P_2, P_3 be the intersection points of the lines AB and ED, AF and CD, CB and EF, respectively. Then the points P_1, P_2, P_3 are collinear.*

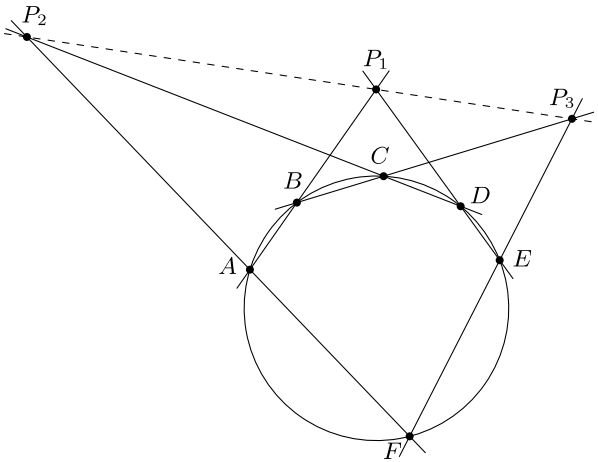


Figure 12.7. Pascal's theorem.

The theorem is illustrated by Figure 12.7, in which the conic is a circle. In fact, Pascal actually proved the theorem in this particular case without any loss of generality – he knew all conics are projectively equivalent to the circle. Here we do not present the (not very difficult) proof of his theorem.

Remark 12.7.4. Note that the theorem is true in $\mathbb{R}P^2$ as well as in $\mathbb{R}^2$. To formulate it in full generality as a Euclidean theorem, one has to consider several singular cases (which arise when one of the points P_i "goes to infinity"); in these cases the proof differs somewhat from the proof in the generic case. Note also that the Euclidean versions have metric proofs (see Problem 12.14), but the projective proof is, in a sense, more natural. Similar remarks hold for the Pappus and the Desargues theorems.

12.8. Projective space $\mathbb{R}P^3$

In this section we very briefly describe three-dimensional projective geometry.

12.8.1. Definition of projective space. The projective space $\mathbb{R}P^3$ can be defined in terms of homogeneous coordinates as explained in Subsection 12.2.3, but here we adopt a more geometric approach. Namely, we consider four-dimensional Euclidean space $\mathbb{R}^4$ and for the *points* of $\mathbb{R}P^3$ take the straight lines passing through the origin O of $\mathbb{R}^4$ and define the transformation group Proj(3) of $\mathbb{R}P^3$ as in the two-dimensional case (using GL(4) instead of GL(3)). We then define the *lines* of $\mathbb{R}P^3$ as the planes passing through the origin O and its *planes* as the three-dimensional hyperplanes of $\mathbb{R}^4$ passing through O.

The following basic statements immediately follow from the above definitions.

I. *One and only one "line" passes through any two distinct "points".*

II. *Any two distinct "planes" intersect in one and only one "line".*

Thus there are no parallel lines or parallel planes in this geometry. Moreover, there is no natural distance function in $\mathbb{R}P^3$ compatible with its geometry, and so no measure of areas or angles, and no perpendiculars.

12.8.2. Properties of projective transformations. Without going into detail, let us just mention that there is a "five point theorem" similar to the "four point theorem" 12.3.3 and that the cross ratio of four collinear points is invariant under projective transformations. There is a neat theory of quadrics (surfaces given by second degree equations) in which, for example, the hyperboloid of two sheets is (projectively) equivalent to the hyperboloid of one sheet and to the ellipsoid.

12.8.3. Projective duality in space. Just as in $\mathbb{R}P^2$, in $\mathbb{R}P^3$ there is a *duality principle*, but a somewhat more sophisticated one: it involves not only points and lines, but also planes. After replacing the expressions "passing through", "intersecting in", etc. by appropriate versions of the notion of incidence and using the expressions "copunctal" and "coplanar" in the formulation of a theorem, we obtain the *dual theorem* simply by interchanging the words "point" and "plane" (and not changing the word "line", since lines are self-dual). The dual theorem will also be correct, since its proof can be obtained by "dualizing" the proof of the original theorem. For example, the properties I and II are dual to each other.

12.9. Problems

12.1. Five distinct collinear points A, B, C, D, E are given. Prove that

$$\langle A, B, C, D\rangle \cdot \langle A, B, D, E\rangle \cdot \langle A, B, E, C\rangle = 1.$$

12.2. How many different values does the cross ratio of four points on a line take when the order of the points is changed?

12.3. Calculate the cross ratio of the following four points lying on the infinite line Λ_∞: $(x_i : y_i; 0)$, $i = 1, 2, 3, 4$.

12.4. Prove Theorem 12.4.3.

12.5. Four planes pass through a common line l, while the line m intersects all four planes. Prove that the cross ratio of the intersection points of m with the planes does not depend on the choice of m.

12.6. State and prove the theorem dual to the Pappus theorem. Draw the corresponding picture.

12.7. State and prove the theorem dual to Desargues' theorem. Draw the corresponding picture.

12.8*. Prove that under projective duality any point on a conic is taken to a line tangent to the dual conic.

12.9. Using Problem 12.8, state and prove the theorem dual to Pascal's theorem (the dual theorem is known as *Brianchon's Theorem*). Draw the corresponding picture.

12.10. Three skew lines l, l_1, l_2 in $\mathbb{R}^3$ are given. To a point $A_1 \in l_1$ let us assign the point A_2 at which the line l_2 intersects the plane determined by A_1 and l. Prove that the assignment $A_1 \mapsto A_2$ is a projective map of l_1 onto l_2.

12.11. The lines $l_1, \ldots, l_{n-1}$ and l are given on the plane. The points $O_1, \ldots, O_n$ are chosen on l. The lines containing the sides of a polygon $A_1, \ldots, A_n$ pass through the points $O_1, \ldots, O_n$ while its vertices $A_1, \ldots, A_{n-1}$ move along the lines $l_1, \ldots, l_{n-1}$. Prove that the vertex A_n also moves along a straight line.

12.12. Prove the triangle inequality for the hyperbolic metric by using appropriate projective transformations.

12.13. Prove the Euclidean version of Pascal's theorem for the case of the circle.

Chapter 13

"Projective Geometry Is All Geometry"

The title of this chapter is a quotation from Arthur Cayley, the outstanding 19th century British mathematician, one of the founders of projective geometry. The aim of this chapter is to give a precise mathematical meaning to these words, namely to show that the three principal continuous geometries, parabolic (Euclid), hyperbolic (Lobachevsky), and elliptic (Riemann), are *subgeometries* of projective geometry. We will prove this in dimension two, i.e., show that the projective plane "contains" (in a certain precise sense) the hyperbolic plane, the elliptic plane, and the Euclidean plane. Since the discrete geometries that we also studied in this book are, in turn, subgeometries of the three principal continuous ones, this means that all the geometries studied so far in this course are parts of projective geometry.

But first we recall the notion of subgeometry, which appeared briefly in Chapter 1.

13.1. Subgeometries

13.1.1. Recall that two geometries $(X : G)$ and $(Y : H)$ are isomorphic if there is an equivariant bijection between them, i.e., a bijection

between their sets of points and an isomorphism between their transformation groups which are compatible (for the detailed definition, see Chapter 1). Further, the geometry $(X : G)$ is a *subgeometry* of $(Y : H)$ if there is an injective map $i : X \to Y$ and a monomorphism $\gamma : G \to H$ compatible with the group actions, i.e., satisfying $(i(x))(\gamma(g)) = i(xg)$. (In this formula, we use the notation xg for the result of the action of the element $g \in G$ on the point $x \in X$; thus $(i(x))(\gamma(g))$ stands for the result of the action of the element $\gamma(g) \in H$ on the point $i(x) \in Y$.)

Of course any geometry isomorphic to the given one is its subgeometry, but we are interested in the case when it is a *proper* subgeometry, i.e., when i is not a bijection, or γ is not an isomorphism, or both.

13.1.2. Here are two toy examples of proper subgeometries:

• the motion group of the regular dodecagon (regular polygon of 12 sides) is a subgeometry of the dodecahedron with dihedral group $\mathbb{D}_{12}$ acting on it;

• the dihedral group $\mathbb{D}_6$ acting on the regular dodecahedron defines a subgeometry of Euclidean plane geometry $(\mathbb{R}^2 : \mathrm{Ismtr}_d(\mathbb{R}^2))$. (Here and below, instead of $\mathrm{Sym}(\mathbb{R}^2)$ we use the notation $\mathrm{Ismtr}_d(\mathbb{R}^2)$ or $\mathrm{Ismtr}(\mathbb{R}^2)$ for the isometry group of Euclidean space w.r.t. the standard metric.)

13.2. The Euclidean plane as a subgeometry of the projective plane $\mathbb{R}P^2$

13.2.1. The fact that the Euclidean plane $(\mathbb{R}^2 : \mathrm{Isom}(\mathbb{R}^2))$ is a subgeometry of the projective plane $(\mathbb{R}P^2 : \mathrm{Proj}(2))$ is rather obvious if we interpret $\mathbb{R}P^2$ (in the homogeneous coordinate model, see Section 11.2) as the plane

$$\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_3 = 1\}$$

supplied with the "line at infinity" $\Lambda_\infty = \{(x_1 : x_2 : x_3) \mid x_3 = 0\}$, i.e., if we take $\mathbb{R}P^2 = \Pi \cup \Lambda_\infty$.

Indeed, let us define $i : \mathbb{R}^2 \to \mathbb{R}P^2 = \Pi \cup \Lambda_\infty$ in the obvious way, i.e., by setting $i((x_1, x_2)) := (x_1, x_2, 1)$, and define $\gamma : \mathrm{Ismtr}(\mathbb{R}^2) \to \mathrm{GL}(3)$ as follows. Let $g \in \mathrm{Ismtr}(\mathbb{R}^2)$, let $(\overrightarrow{AB}, \overrightarrow{AC})$ be an orthonormal frame in $\mathbb{R}^2$ and $(\overrightarrow{A'B'}, \overrightarrow{A'C'})$ its image under g. For $\gamma(g)$ we take the element of $\mathrm{Proj}(2)$ that takes the three lines OA, OB, OC to the three lines OA', OB', OC'. This construction is shown in the figure.

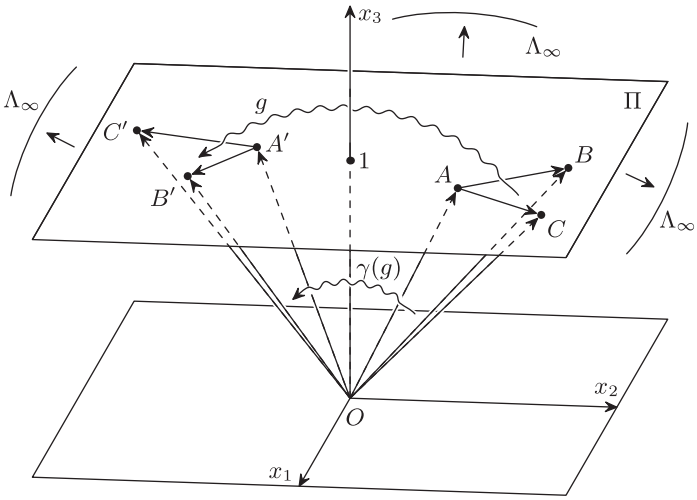


Figure 13.1. The Euclidean plane as a subgeometry of $\mathbb{R}P^2$.

Theorem 13.2.2. *The construction described above shows that the Euclidean plane is a subgeometry of the projective plane.*

Proof. The theorem is obvious: clearly, i is injective, γ is a monomorphism, and the fact that compatibility holds is also immediate. □

13.3. The hyperbolic plane as a subgeometry of the projective plane $\mathbb{R}P^2$

13.3.1. The fact that the hyperbolic plane $(\mathbb{H}^2 : M)$ is a subgeometry of the projective plane $(\mathbb{R}P^2 : \mathrm{Proj}(2))$ is best seen by using the Cayley–Klein model and interpreting $\mathbb{R}P^2$ (as in Section 12.2 of the

preceding chapter) as the plane

$$\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 1\}$$

supplied with the "line at infinity" $\Lambda_\infty = \{(x_1 : x_2 : x_3) \mid x_3 = 0\}$, i.e., by taking $\mathbb{R}P^2 = \Pi \cup \Lambda_\infty$.

We recall that the Cayley–Klein model was defined as $(\mathbb{H}^2 : \mathrm{Ismtr}_\lambda(\mathbb{H}^2))$, where $\mathbb{H}^2$ is the unit open disk and λ is the metric given by the formula $\lambda(A, B) = (1/2)|\ln(\langle A, B, X, Y\rangle)|$ (for the details, see Section 9.2).

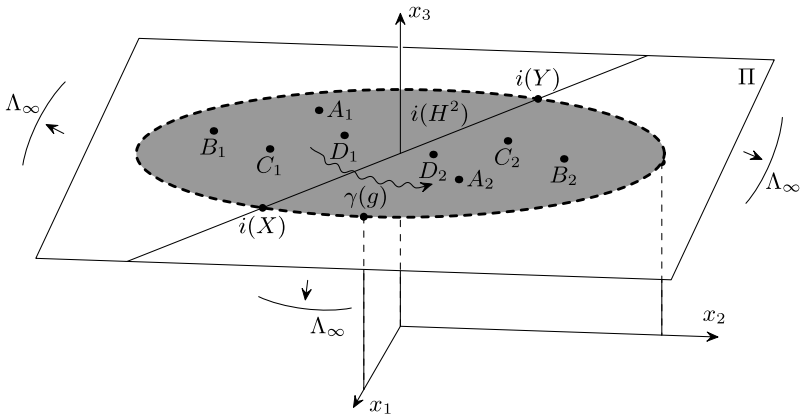


Figure 13.2. The Cayley–Klein model as a subgeometry of $\mathbb{R}P^2$.

Now let us define $i : \mathbb{H}^2 \to \mathbb{R}P^2 = \Pi \cup \Lambda_\infty$ in the obvious way, i.e., by setting $i((x_1, x_2)) := (x_1, x_2, 1)$, and define $\gamma : \mathrm{Ismtr}_\lambda(\mathbb{H}^2) \to \mathrm{Proj}(2)$ as follows. Let $g \in \mathrm{Ismtr}_\lambda(H^2)$. Take four points $A, B, C, D \in \mathbb{H}^2$ in general position and consider their images $Ag, Bg, Cg, Dg \in \mathbb{H}^2$ under g. Denote

$$A_1 = i(A), \quad B_1 = i(B), \quad C_1 = i(C), \quad D_1 = i(D) \in \mathbb{H}^2,$$

$$A_2 = i(Ag), \quad B_2 = i(Bg), \quad C_2 = i(Cg), \quad D_2 = i(Dg) \in \mathbb{H}^2.$$

The two quadruples of points $A_i, B_i, C_i, D_i \in \Pi$, $i = 1, 2$, are in general position, and so by Theorem 12.3.3 there exists a unique projective transformation taking A_1, B_1, C_1, D_1 to A_2, B_2, C_2, D_2; we take this transformation to be $\gamma(g)$. The construction is shown in the figure.

Basically, this construction is simply the natural extension of the action of g from the open unit disk

$$\{(x_1, x_2, 1) \mid x_1^2 + x_2^2 < 1\} = i(\mathbb{H}^2)$$

to the entire projective plane. To any "straight line" of $\mathbb{H}^2$ (i.e., any chord XY of the unit circle) corresponds the straight line joining the points $i(X)$, $i(Y)$ in the projective plane; to parallel or nonintersecting lines in $\mathbb{H}^2$ (chords of the unit circle) correspond straight lines in $\mathbb{R}P^2$ that actually intersect (at a point outside the disk $i(\mathbb{H}^2)$, possibly on the "infinite line" Λ_∞).

Theorem 13.3.2. *The construction described above shows that the hyperbolic plane is a subgeometry of the projective plane.*

Proof. The map i is obviously injective, so that it remains to show that the restriction of $\gamma(g)$ to the open disk

$$\{(x_1, x_2, 1) \mid x_1^2 + x_2^2 < 1\} = i(\mathbb{H}^2)$$

coincides with γ. This is a consequence of the fact that projective transformations preserve the cross ratio of any four collinear points, and therefore preserve the distance λ between points inside $i(\mathbb{H}^2)$ (λ being the absolute value of the logarithm of the appropriate cross ratio). But g is an isometry (with respect to λ), it coincides with the restriction of $\gamma(g)$ to $i(\mathbb{H}^2)$ on three noncollinear points, therefore it coincides with this restriction on all of $i(\mathbb{H}^2)$. This proves the theorem. □

Remark 13.3.3. It can be proved that the subgroup of Proj(2) that takes the circle $\{(x_1, x_2, 1) \mid x_1^2+x_2^2 = 1\}$ to itself is actually isomorphic to $\mathrm{Ismtr}_\lambda(\mathbb{H}^2)$, and this isomorphism is often used to establish various formulations expressing the fact that the hyperbolic plane is "a part of" the projective plane. One can also define (a model of) hyperbolic geometry by using this fact. We do not need this fact in our approach to this topic, so we omit its proof here.

13.4. The elliptic plane as a subgeometry of $\mathbb{R}P^2$

13.4.1. As in the previous two sections, we regard $\mathbb{R}P^2$ as the plane Π with the line at infinity Λ_∞ added to it. Our model of Riemannian

two-dimensional elliptic geometry $\mathbb{E}\mathrm{ll}^2$ will be the standard one, i.e., the unit sphere with its antipodal points identified, namely: $\mathbb{E}\mathrm{ll}^2 = (\mathbb{S}^2/\mathrm{Ant} : \mathrm{O}(3))$. We think of this sphere as lying on the plane Π, touching it at the point $(0, 0, 1)$.

We first construct the inclusion (which will actually be a bijection) of $\mathbb{S}^2/\mathrm{Ant}$ to $\mathbb{R}P^2$ by simply projecting it from the center of the sphere onto $\Pi \cup \Lambda_\infty$. Note that "straight lines" in $\mathbb{S}^2/\mathrm{Ant}$ (i.e., great circles of the sphere with diametrically opposed points identified) will be mapped to straight lines of the projective plane; in particular, the equator of the sphere will be mapped to the "infinite line" Λ_∞. Note also that spherical triangles (not intersecting the equator) will be projected to ordinary rectilinear triangles in Π, but their angles will not be preserved.

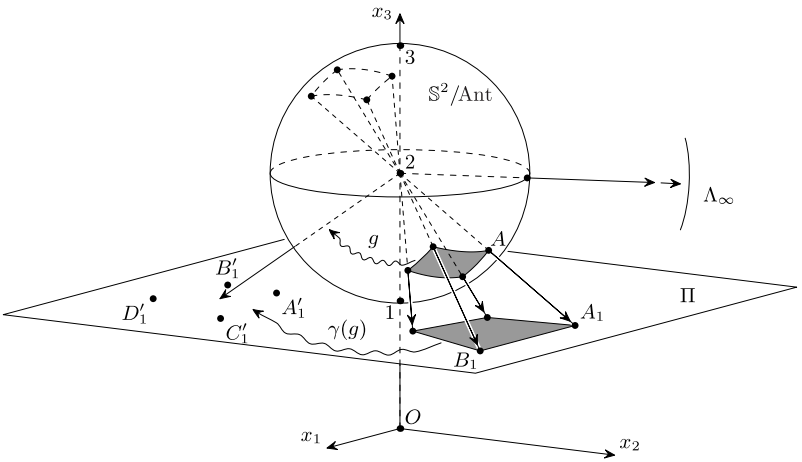


Figure 13.3. Bijection between the elliptic plane and $\mathbb{R}P^2$.

To construct the monomorphism $\gamma : \mathrm{O}(3) \to \mathrm{Proj}(2)$, we choose two perpendicular arcs AB and AC (that do not intersect the equator) and denote by BCD the triangle symmetric to triangle ABC with respect to the line BC. Denote by A_1, B_1, C_1, D_1 the central projections of the points A, B, C, D to the plane Π. Now suppose $g \in \mathrm{O}(3)$ takes the points A, B, C, D to A', B', C', D', and denote by

A_1', B_1', C_1', D_1' their projections to Π. We define $\gamma(g)$ as the projective transformation that takes A', B', C', D' to A_1', B_1', C_1', D_1' (such a projection exists and is unique by Theorem 12.3.3). The construction is shown in Figure 13.3.

Theorem 13.4.2. *The construction described above shows that the Riemannian elliptic plane is a subgeometry of the projective plane.*

Proof. The theorem is an easy consequence of the following lemma, whose proof is the object of Problem 13.3.

Lemma 13.4.3. *The map γ described above is a monomorphism of* O(3) *to* Proj(2).

Indeed, the monomorphism γ is compatible with the map i by construction, so the theorem follows. □

13.5. Problems

13.1. Prove that any projective transformation of the projective plane $\mathbb{R}P^2$ preserves the cross ratio of collinear points

13.2. Prove that, conversely, any transformation of the projective plane that preserves the cross ratio of all collinear points is projective.

13.3. Prove Lemma 13.4.3.

13.4. Give an example of a spherical triangle whose angle sum is close to 2π and describe its image under the central projection defined in Section 12.4.

13.5. Show that for any $\varepsilon > 0$ and any positive number S, there exists a spherical triangle of area less than ε whose image under the central projection defined in Section 13.4 is of area greater than S.

13.7. Prove that the subgroup of projective transformations that take the unit circle centered at the origin to itself is isomorphic to the isometry group of the hyperbolic plane.

13.8. Generalize and solve the previous problem by replacing the circle by an arbitrary oval (nondegenerate second degree curve).

Chapter 14

Finite Geometries

A finite geometry is geometry whose set of points is finite. In that situation, the possibilities for the transformation group are extremely varied, and Klein's definition of geometry is too general to single out those finite geometries that actually deserve to be called geometries. Thus one must impose restrictions on the group actions involved, and this can be done by using coordinates from linear spaces over finite fields. Another approach involves introducing the notion of "straight line" and imposing conditions (axioms) which make the geometries "projective" or "affine" in a certain sense.

Unfortunately, the two approaches are not equivalent, the axiomatic approach yielding a wider class of finite planes than the algebraic coordinate one. However, it turns out that the two approaches *are* equivalent if and only if Desargues' theorem holds in the finite geometry considered.

It should be noted that some basic natural questions about finite geometries are at present unanswered and that these geometries are the object of active ongoing research. Some of these questions and related conjectures are mentioned in Section 14.11.

14.1. Small finite geometries

In this section, we try to classify all the geometries with a "small" number of points. By classifying we mean listing (without repetitions) all the geometries with a given number of points $k := |X|$ up to isomorphism. Recall that two geometries are isomorphic if there is an equivariant bijection between them, i.e., a bijection between their sets of points and an isomorphism between their transformation groups which is compatible with the bijection (for the detailed definition, see Chapter 1).

There is of course only one geometry with one point. For $|X| = 2$ there are two geometries (with $|G| = 2$ and $|G| = 1$). For $|X| = 3$ there are four: the symmetries (= isometries) of the vertices of the equilateral triangle ($G = \mathbb{S}_3$), the motions of the vertices of the equilateral triangle ($G = \mathbb{Z}_3$), the symmetries of the vertices of the isosceles triangle ($G = \mathbb{Z}_2$). For $|X| = 4$ there are ten: the symmetries of the regular tetrahedron, its motions, the symmetries of the square, its motions, the rotations of the square by 0 and π, the symmetries of the rhombus, and four more geometries obtained when the transformation group has a fixed point (the same one for each element).

For $|X| \geq 5$ the situation becomes too complicated to handle, while for $|X| \geq 10$, even a supercomputer is powerless.

To continue our study, we need to specify some reasonable narrower classes of finite geometries. To do that, we need some algebra.

14.2. Finite fields

The modern logical foundation of ordinary Euclidean affine geometry is the notion of vector space over the real number field. To construct something similar in the finite case, we need *finite fields*.

Theorem 14.2.1. *For any $q = p^m$, where p is prime and m is a positive integer, there exists exactly one (up to isomorphism) field consisting of q elements, called the finite field of order q and denoted by $\mathbb{F}(q)$. There are no other finite fields.*

We will not prove this theorem (the proof belongs to algebra courses) and only present the simplest nontrivial example, the field $\mathbb{F}(4) = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, by showing its addition and multiplication tables:

$+$	**0**	**1**	**2**	**3**
0	**0**	**1**	**2**	**3**
1	**1**	**0**	**3**	**2**
2	**2**	**3**	**0**	**1**
3	**3**	**2**	**1**	**0**

$\times$	**0**	**1**	**2**	**3**
0	**0**	**0**	**0**	**0**
1	**0**	**1**	**2**	**3**
2	**0**	**2**	**3**	**1**
3	**0**	**3**	**1**	**2**

In order to get a feeling for the structure of the fields $\mathbb{F}(q)$, we invite the reader to construct the addition and multiplication tables for, say, $\mathbb{F}(3^2)$.

14.3. Example: the finite affine plane over $\mathbb{F}(5)$

In this section we will construct a finite affine plane geometry starting from the finite field $\mathbb{F}(p)$, where p is a prime number (i.e., in the case $m = 1$). To make the construction more concrete, we will carry it out for $p = 5$, although it works for any prime p.

14.3.1. Let us define the *affine plane* $A\mathbb{F}(5)$ of order 5 as the set $\{(x, y) \mid x \in \mathbb{F}(5), y \in \mathbb{F}(5)\}$ of pairs (coordinates of *points*). As in ordinary Euclidean geometry, two points $T = (a, b)$, $S = (c, d)$ determine a *vector* $\overrightarrow{TS} = \{c - a, d - b\}$. We will define *straight lines* as in analytic geometry, i.e., by setting $A(t) = A_0 + t\overrightarrow{v}$, where A_0 is a point, $\overrightarrow{v}$ is a vector, and t runs over $\mathbb{F}(5)$. For example, if we take $A_0 = (0, 0)$ and $\overrightarrow{v} = (1, 2)$, we obtain the "straight line"

$$\big\{(0,0), (1,2), (2,4), (3,1), (4,3)\big\}.$$

Thus we obtain a total of 30 straight lines, 25 points, 5 points on each line, and 6 lines passing through each point. In Figure 14.1, we have shown the six lines passing through the point $(0, 0)$.

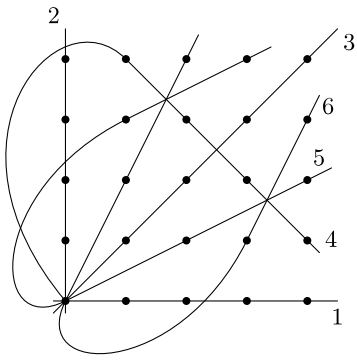


Figure 14.1. Six lines in $A\mathbb{F}(5)$.

Arguing in the same way in the general case, we obtain $p^2 + p$ straight lines, p^2 points, p points on each line, and $p+1$ lines passing through each point.

14.3.2. The same result can be obtained by using the orbit space of an appropriate geometry. Let $\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ be the integer lattice on the plane and let $(\mathbb{Z} \oplus \mathbb{Z} : G)$ be the geometry defined by the transformation group G, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, acting by coordinate shifts by 5, i.e.,

$$G \ni (k, l) : (m, n) \mapsto (m + 5k, n + 5l).$$

The orbit space of this action consists of 25 "points". We identify them with the 25 points of the lattice with nonnegative coordinates less than 5. The "straight line" passing through two points of this 5 by 5 square is defined as follows: construct the Euclidean line joining these two points in $\mathbb{R}^2$, take all the integer points on this line and reduce both of their coordinates mod 5, obtaining three more points in the square; together with the two given points, they constitute a "straight line".

Geometrically, you can visualize this as the covering of the torus by the plane: under this map the points of the square lattice are "wrapped around" the 25 points on the torus.

14.4. Example: the finite affine plane over $\mathbb{F}(2^2)$

We now start our constructions with the field $\mathbb{F}(2^2)$. Define the *affine plane* over $\mathbb{F}(4)$ as the set $\{(x, y) \mid x \in \mathbb{F}(4), y \in \mathbb{F}(4)\}$ of pairs (coordinates of *points*). Using the same approach as in Section 14.3 (including the "vector definition" of straight lines), let us consider the line passing through the point $(0, 0)$ with direction vector $\{1, 1\}$. This "line" has only two points $(0, 0)$ and $(1, 1)$ because $(1, 1) + \{1, 1\} = (0, 0)$. Thus the coordinate approach does not work over $\mathbb{F}(4)$.

Nevertheless, a reasonable affine geometry with 4 points on each line can be constructed on the set of points P by defining straight lines in a different way. In particular, the "straight line" that passes through the points $(0, 0)$ and $(1, 1)$ also contains the points $(2, 2)$ and $(3, 3)$, while the line passing through $(0, 0)$, $(1, 2)$ contains the points $(2, 3)$ and $(3, 1)$. In this geometry, there are $16 = q^2$ points, $20 = q^2 + q$ straight lines, $4 = q$ points on each line, and $5 = q + 1$ lines pass through each point. The five lines passing through the point $(0, 0)$ are shown in Figure 14.2.

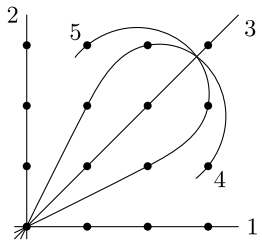


Figure 14.2. Five lines in $A\mathbb{F}(4)$.

The result is a geometry called the *finite affine plane* over the field $\mathbb{F}(4)$ and is denoted by $A\mathbb{F}(4)$. The set $A\mathbb{F}(4)$ is indeed a geometry in the sense of Klein, $(A\mathbb{F}(4) : \Gamma)$, if for the transformation group Γ we take the set of all bijections of $A\mathbb{F}(4)$ that map lines into lines.

In the general case, i.e., when $\mathbb{F} = \mathbb{F}(q)$, $q = p^m$, $m > 1$, with prime p, one can also construct the finite affine plane $A\mathbb{F}(q)$, but the

direct construction is rather tedious, and we omit it. However, we will present a neat indirect construction via finite projective geometries in Section 14.8.

First, we give an example of a finite projective geometry.

14.5. Example of a finite projective plane

14.5.1. Let $A\mathbb{F}(4)$ be the finite affine plane for $q = 2^2$. We say that two lines of $A\mathbb{F}(4)$ are *parallel* if they coincide or have no common points. Parallelism is an equivalence relation, and so all lines are partitioned into equivalence classes of parallel lines. It is easy to see that there are 5 such classes. To $A\mathbb{F}(4)$ let us add 5 points (called *points at infinity*) and agree that they all lie on one straight line (the *line at infinity*). The set thus obtained is called the *projectivization* of the affine plane $A\mathbb{F}(4)$ and is denoted by $P\mathbb{F}(4)$; it has 21 points, 21 straight lines, 5 points on each line, 5 lines passing through each point, and any two distinct lines have exactly one common point. The projective plane $P\mathbb{F}(4)$ is shown in Figure 14.3.

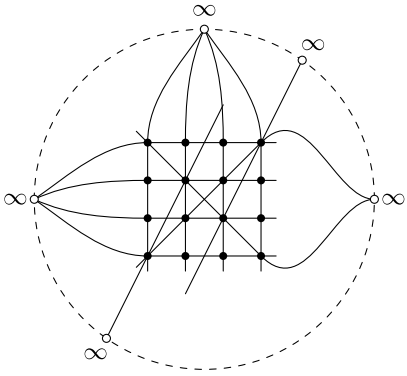


Figure 14.3. Projectivization of $A\mathbb{F}(4)$.

14.5.2. The construction described above for $q = 4$ actually works for any $q = p^m$ with prime p. One obtains the projective geometry $P\mathbb{F}(q)$; it has $q^2 + q + 1$ points, $q^2 + q + 1$ straight lines, $q + 1$ points on each line, $q + 1$ lines passing through each point.

14.6. Axioms for finite affine planes

14.6.1. A more traditional approach to finite geometries is the axiomatic approach. A *finite affine plane* is a nonempty finite set of elements P (called *points*) with a family L of subsets (called *lines*) that satisfy the axioms:

Aff.1. *There is exactly one line passing through any two distinct points.*

Aff.2. *There is exactly one line parallel to a given line and passing through a given point.* (Two lines are called *parallel* if they have no common points or if they coincide.)

Aff.3. *There exists a generic triangle* (*three points not belonging to one and the same line*).

Here the second axiom ensures that the dimension of the set of points is less than or equal to 2. The third axiom ensures that its dimension is greater than or equal to 2. Thus the dimension of the set of points is two, and this set can be regarded as a "plane". The construction of the two simplest affine planes (with 4 and 9 points) is the object of Problem 14.1.

Theorem 14.6.2. (i) *For every $q = p^m$, where p is prime and m is a positive integer, there exists an affine geometry $\mathsf{P} = A\mathbb{F}(q)$ with q points on a line.*

(ii) *The geometry $\mathsf{P} = A\mathbb{F}(q)$ has q^2 points, a family of $q^2 + q$ subsets L that satisfies the axioms Aff.1–Aff.3.*

(iii) *If Γ_q is the group of bijections of P that map lines (i.e., elements of L) into lines, then (P, Γ_q) is a geometry in the sense of Klein called an affine Galois plane of order q.*

The existence of $A\mathbb{F}(q)$ (item (i) of the theorem) will be proved in 14.8.3 below. The proof of items (ii)–(iii) is a series of moderately difficult problems for the reader (14.2–14.6) that appear in the problem section.

14.7. Axioms for finite projective planes

14.7.1. A finite projective plane is a nonempty finite set of elements P (called *points*) with a family L of subsets (called *lines*) that satisfy the following axioms:

Proj.1. *There is exactly one line passing through a given pair of distinct points.*

Proj.2. *There is exactly one point contained in a given pair of distinct lines.*

Proj.3. *There exist four points that determine six distinct lines.*

Proj.4. *There exist four lines that determine six distinct points.*

Actually the fourth axiom is redundant (it follows from the first three), we include it for the sake of symmetry.

The simplest finite projective plane (called the *Fano plane*) is shown in Figure 14.4. It has 7 points, 7 lines, 3 points on each line, and 3 lines passing through each point. The four points in the middle of the picture satisfy the axiom Proj.3. The Fano plane can be constructed from the four point affine plane by adding the "line at infinity", as explained in 14.5.1.

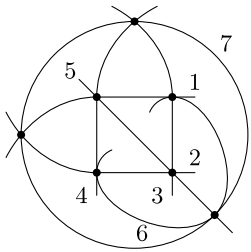


Figure 14.4. The Fano plane.

14.7.2. Projective duality. Just as in the case of the real projective plane $\mathbb{R}P^2$, the finite projective plane satisfies the *Duality Principle: Interchanging the words "point" and "line" in the statement of any theorem and accordingly modifying the wording of the incidence relations, one obtains another theorem.* This principle follows from

the fact that the four axioms split into two pairs dual to each other. However, the finite projective plane obtained from a given one by duality is not necessarily isomorphic to the given one. Questions of duality are rather delicate in the finite case, and we do not discuss them here.

Theorem 14.7.3. *If* (P, L) *is a finite projective plane, then there exists a natural number* n*, called the order of the plane, such that:*

(i) *each line contains* $n+1$ *points;*

(ii) *each point is contained in* $n+1$ *lines;*

(iii) *the number of points is equal to the number of lines and equal to* n^2+n+1.

Remark 14.7.4. The theorem does not assert the existence of finite projective planes: in it, it is *assumed* that a finite projective plane is given, and thus it only asserts that if a plane satisfying axioms Proj.1–Proj.3 exists, then its number of lines and points satisfies the constraints (i)–(iii).

14.7.5. Proof of Theorem 14.7.3. Suppose (P, L) is a finite projective plane, $l, m \in \mathsf{L}$, and let $a \in \mathsf{P}$ be a point not lying on l nor on m (such a point exists by axiom Proj.3). Consider the map f of the set of points of the line l to the set of points of m that assigns to each point $x \in l$ the intersection point of the lines xa and m. Axioms Proj.1–Proj.2 imply that f is well defined and bijective. Denoting the number of points on l by $n+1$, we see that item (i) is proved. Item (ii) follows by the duality principle. To prove (iii), fix some point $a \in \mathsf{P}$. Each line passing through a passes through n other points, and so $|\mathsf{P}| = (n+1)n+1 = n^2+n+1$. By duality we have $|\mathsf{L}| = n^2+n+1$, which concludes the proof. □

14.7.6. Remarks. (1) One can pass from the finite affine plane to the projective plane by adding $q+1$ "points at infinity" (corresponding to each class of parallel lines) and one new line (the line of all points at infinity). Conversely, one can pass from a projective plane to an affine plane by removing one line (with all its points). Unfortunately, the result is not well defined: it may depend on the choice of the line!

(2) There is no uniqueness theorem for projective planes of order p^m for $m > 1$ (for example, there is are several nonisomorphic projective planes of order 9, see Problems 14.9).

(3) It is not known at present for what values of q there exist projective planes of order q. Specifically, this question is unanswered already for $q = 12$. This question, and other open questions, as well as related conjectures, are briefly discussed in Section 14.11.

14.8. Constructing projective planes over finite fields

In this section, we give a constructive definition of the finite projective planes based on linear spaces over finite fields, similar to the definition of the real projective plane $\mathbb{R}P^2$ (cf. 12.1).

14.8.1. Main construction. Consider the three-dimensional vector space V over the finite field $\mathbb{F} = \mathbb{F}(p^m)$, where p is prime. Denote by P the set of one-dimensional subspaces of V, which we now call *points*, and by L the set of two-dimensional subspaces, which we now call *lines*; we say that a line $l \in \mathsf{L}$ *passes through* a point $p \in \mathsf{P}$ (or p is contained in l, or l *contains* p) if we have the inclusion of linear spaces $p \subset l$.

Theorem 14.8.2. (i) *The construction described above yields a finite projective plane* (P, L) *of order* $q = p^m$.

(ii) *If we define the transformation group of* P *as the set of bijections* Γ *of* P *that take lines to lines, then* (P, Γ) *is a geometry in the sense of Klein.*

Proof. All four axioms Proj.1–Proj.4 are immediate consequences of the main construction. Item (ii) is the object of Problem 14.6. □

The geometry thus constructed is called the *finite projective space* over the field $\mathbb{F}(p^m)$ and is denoted by $P\mathbb{F}(p^m)$.

Corollary 14.8.3. *The finite affine plane of order* $q = p^m$, *where* p *is any prime and* m *is any natural number, exists.*

Proof. To construct the required plane, it suffices to remove one line (and all its points) from the finite projective plane of order q. □

14.9. The Desargues theorem

The Desargues theorem, which we proved for the real projective plane $\mathbb{R}P^2$, is not true for arbitrary finite projective planes. However, we have the following statement.

Theorem 14.9.1. *The Desargues theorem holds for the finite projective planes $P\mathbb{F}(p^m) = (\mathsf{P}, \mathsf{L})$, i.e., three lines $x_1y_1, x_2y_2, x_3y_3 \in \mathsf{L}$ intersect at one point if and only if the intersection points $z_1, z_2, z_3 \in \mathsf{P}$ of the pairs of lines x_2x_3 and y_2y_3, x_3x_1 and y_3y_1, x_1x_2 and y_1y_2, respectively, are collinear.*

Proof. In the proof, we will use the model of $P\mathbb{F}(p^m)$ given by the construction 14.8.1, i.e., we regard points as one-dimensional linear subspaces of the vector space over $\mathbb{F}(p^m)$ and lines as two-dimensional subspaces.

First let us note that the Desargues theorem is self-dual, and therefore it suffices to prove the "only if" part, i.e., assuming that the lines A_1B_1, A_2B_2, A_3B_3 intersect at one point (which we denote by S), to show that the intersection points P_1, P_2, P_3 are collinear. If the point S lies in each of the three lines P_1P_2, P_2P_3, P_3P_1, then there is nothing to prove, so we can assume that $S \notin P_2P_3$.

In our model the points S, A_i, B_j, P_k are actually one-dimensional linear spaces, and we will use the same lower case letters s, a_i, b_j, p_k to denote nonzero vectors belonging to (and therefore determining) the corresponding linear spaces.

Now since the vectors s, a_1, b_1 belong to the same two-dimensional space, they are linearly dependent, and (by an appropriate choice of these vectors in their linear spaces) we can write $b_1 = a_1 + s$. It is easy to see that the vectors a_1, p_2, p_3 are linearly independent, and therefore we can put

$$b_1 = \alpha_1 a_1 + \alpha_2 p_2 + \alpha_3 p_3,$$

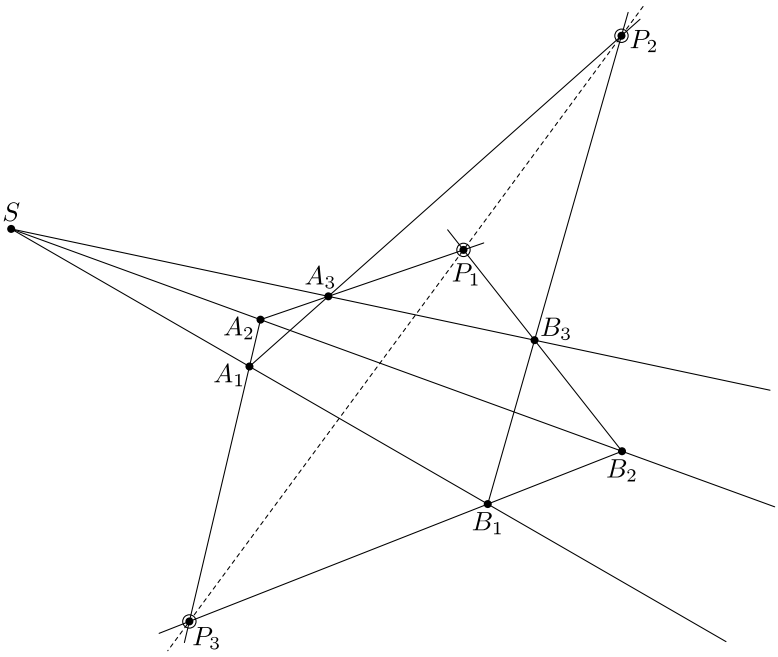


Figure 14.5. Desargues' theorem.

where $\alpha_1, \alpha_2, \alpha_3 \in F$. Consider the linear operator φ on V given by

$$\varphi(a_1) = b_1, \quad \varphi(p_2) = p_2, \quad \varphi(p_3) = p_3.$$

Then we have

$$\begin{aligned}(14.1)\quad \varphi(s) = \varphi(b_1 - a_1) &= (\alpha_1 - 1)b_1 + \alpha_2 p_2 + \alpha_3 p_3 \\ &= (\alpha_1 - 1)b_1 + b_1 - \alpha_1 a_1 = \alpha_1 s.\end{aligned}$$

The linear operator φ is nondsingular, so it takes linear subspaces to linear subspaces of the same dimension. In particular, we have

$$\varphi(A_1) = B_1,\ \varphi(P_2) = P_2,\ \varphi(P_3) = P_3,\ \varphi(S) = S.$$

The vectors p_2, p_3 form a basis of the line P_2P_3 (regarded as a two-dimensional vector space), and so the operator φ is the identity on this line. Now if Λ is any line passing through S, then, since φ

leaves S in place as well as the intersection point of the lines Λ and P_2P_3, it follows that $\varphi(\Lambda) = \Lambda$.

Now the point A_2 lies on the lines SA_2 and B_1P_3, and therefore $\varphi(A_2)$ is the intersection point of the lines SA_2 and B_1P_3, and so $\varphi(A_2) = B_2$. Similarly, $\varphi(A_3) = B_3$. Thus $\varphi(A_2A_3) = B_2B_3$. Now let P be the intersection point of the lines A_2A_3 and P_2P_3. Then the point $\varphi(P)$ lies on the line B_2B_3 and, at the same time, $\varphi(P) = P$. Therefore, $P = P_1$ and P_1 lies on the line P_2P_3, which was to be proved. □

14.9.2. Remark. Note that this proof (like the proof given in 12.7.1) is, in a certain sense, "three-dimensional": when we replaced points by vectors in the above proof, we were essentially adding a point (the origin of coordinates in the three-dimensional space over $\mathbb{F}(p^m)$) lying outside of the plane containing all the given points.

14.10. Algebraic structures in finite projective planes

Until now, we have been using algebra (finite fields) to construct geometric objects (finite affine and projective planes). Now we will try to move in the opposite direction, i.e., analyze what the geometric axioms for the finite projective plane imply concerning the algebraic structure of the projective line. Unfortunately, it will turn out that the natural and optimistic expectation that axioms Proj.1–Proj.4 imply that there are $p^m + 1$ points on each line (for some prime p and natural number m) and that these points can be added and multiplied in a natural way, thereby forming a field isomorphic to $\mathbb{F}(p^m)$, does not come true. The situation is much more complicated; in the general case one can obtain an algebraic structure from the axioms, but it is not that of a field: its multiplication is not commutative and there is only one distributive law (see 14.10.3 below).

14.10.1. Introducing coordinates. Let (P, L) be a finite projective plane of order $n \geq 2$. (Recall that this means that (P, L) satisfies axioms Proj.1–Proj.4) and one of its lines (and therefore all lines) contains n points. Denote by F a set of n elements; we stress that F is a set of arbitrary symbols, it is not a field; in fact, at first it has no

algebraic operations defined on it. Our aim is to supply F with an algebraic structure (hopefully that of a field) and use it to introduce coordinates in our finite projective plane (P, L).

We begin by choosing two arbitrary elements of F that we denote by 0 and 1. By ∞ we denote a symbol that does not belong to F. Using axiom Proj.3, let us choose an *initial quadrilateral* in our plane, i.e., four points, no three of which lie on one line. Denote these points by $(0,0), (0), (\infty), (1,1)$ and denote the six lines passing through these points as follows:

$$[0,0] := (0,0)(0), \quad [0] := (0,0)(\infty), \quad [\infty] := (0)(\infty),$$
$$[1] := (1,1)(\infty), \quad [0,1] := (1,1)(0), \quad [1,0] := (0,0)(1,1).$$

These six lines intersect in seven points, four of which belong to the initial quadrilateral, and we denote the other three as follows:

$$(1,0) := [1][0,0], \ (0,1) := [0][0,1], \ (1) := [\infty][1,0],$$

where the juxtaposition of two lines determines their intersection, e.g., the formula $(1,0) = [1][0,0]$ means that $(0,1)$ is the intersection point of the lines $[1]$ and $[0,0]$.

If there are no other points in P, then $n = 2$ and it is easy to see that we have obtained the Fano plane. The reader will profit by looking at Figure 14.4 and supplying its points with coordinates as indicated in the construction described above.

If there are other points left, then $n > 2$ and we denote by a an arbitrary element of F other than 0 or 1. For any such a, we define new points and lines by setting

$$[a,0] := (0,0)(a), \ \ (1,a) := [1][a,0], \ \ [0,a] := (0)(1,a),$$
$$(a,a) := [0,a][1,0], \ [a] := (a,a)(\infty), \ (a,0) := [a][0,0], \ (0,a) := [0,a][0].$$

If there are any other elements b in F other than $0, 1, a$, we set

$$(a,b) := [a][0,b], \quad [a,b] := (a)(0,b).$$

Thus we have supplied all the points of our finite projective plane with coordinates, and we know what the intersection points of any two lines are.

14.10.2. Addition and multiplication. Now we can define the sum and product of two arbitrary elements $a, b \in F$ by setting

$$(a, a + b) := [a][1, b], \quad (a, a \cdot b) := [a][b, 0].$$

The motivation behind this definition is that it is compatible with the addition and multiplication induced on points of the projective line in the case of the finite projective plane over the field $\mathbb{F}(p^m)$. The reader is invited to return to the definition of finite projective planes over a field, check that they can be supplied with coordinates as specified above and that the operations defined above coincide with the ones induced by the field $\mathbb{F}(p^m)$.

As we noted before, it is not always true that these operations supply F with a field structure. They satisfy axioms of a structure weaker than that of a field, which we now define.

14.10.3. Almost fields. An *almost field* is a set F with two binary operations, called *addition* and *multiplication*, such that under addition F is an Abelian group with neutral element 0, the set $F \setminus 0$ is a group (not necessarily Abelian) under multiplication and the right distributive law is satisfied, i.e., $(a + b)c = ac + bc$.

When the left distributive law is not satisfied (such examples of almost fields exist), the almost field is not even a ring. We will not describe examples of this type or study almost fields in detail: they are complicated and rather ugly, and we will limit our exposition to the statements (without proofs) of two beautiful theorems and of some open problems.

Theorem 14.10.4. (i) *Given any finite almost field F, a projective plane over F can be determined by using the construction from Section* 14.8 *with F replacing the field $\mathbb{F}(p^m)$.*

(ii) *Given any finite projective plane of order n, there is an almost field F (of order $n - 1$) using which the projective plane can be constructed as indicated in* (i).

The proof of (i) is similar to that in Section 14.8, while (ii) can be proved by a tedious series of geometric constructions needed to verify the numerous axioms of almost fields.

Theorem 14.10.5. *A finite projective plane is a projective plane over the field $\mathbb{F}(p^m)$ if and only Desargues' theorem holds in it.*

The "only if" part was proved above (see 14.8.1), while the "if" part is another complicated series of artificial geometric constructions ensuring the required algebraic axioms.

14.11. Open problems and conjectures

The main open problem here is the following: *For what values of q does there exist a finite projective plane of order q, and for what values of q is the finite projective plane of order q unique?*

We know that there exists one and only one projective plane of the orders 2, 3, 4, 5, 7, 8 (see Problems 14.10–14.11). We also know certain number-theoretic constraints forbidding projective planes of certain orders.

Theorem 14.11.1 (Brack–Raiser)**.** *Let $q \equiv 1$ or $2 \pmod 4$. If there exists a projective plane of order q, then q can be presented as the sum of squares of two natural numbers.*

The proof appearing in the original article by Bruck–Ryser[1] is not easy, and we omit it. This theorem forbids projective planes of orders 6, 14, 21, 22, 30, etc.

Conjecture 14.11.2. *The order q of any finite projective plane is a prime number $q = p$ or a power of a prime $q = p^m$.*

The first natural number q which does not meet the assumptions of the conjecture is 6, and indeed one can prove (see Problem 14.9) that there is no finite projective plane of order 6. The next such number is 10, and it is only in 1991 that it was established, with the aid of a supercomputer, that the conjecture holds there also. But already for $q = 12$ the existence of a projective plane of order q is an open question.

Conjecture 14.11.3. *All the projective planes of prime order p are Desarguian.*

[1] R.H. Bruck, H.J. Ryser, *The non-existence of certain finite projective planes*, Canadian J. Math., Vol. 1 (1949), 88–93.

There are non-Desarguian projective planes of nonprime order. The "smallest" one is of order 9 (Problem 14.16).

14.12. Problems

14.1. Construct an affine geometry having 4 points, and a finite affine geometry having 9 points.

14.2. Suppose that one of the lines of the affine plane (P, L) from Corollary 14.8.3 consists of q points. Prove that the plane P consists of q^2 points.

14.3. Suppose that one of the lines of the affine plane (P, L) from Corollary 14.8.3 consists of q points. Prove that all other lines consist of q points.

14.4. Suppose that one of the lines of the affine plane (P, L) from Corollary 14.8.3 consists of q points. Prove that L consists of $q^2 = q$ lines.

14.5. Suppose that one of the lines of the affine plane (P, L) consists of q points. Prove that $q + 1$ lines pass through each point.

14.6. Prove that the finite affine plane $A\mathbb{F}(p^m)$ is a geometry in the sense of Klein.

14.7. In the affine plane consisting of q^2 points for $q = 3$, construct the system of lines passing through one of the points.

14.8. Describe the projectivization of the affine plane from Problem 14.5.

14.9*. Prove that there does not exist a finite projective plane of order $q = 6$.

14.10. Prove that the projective planes of order 2, 3, 4, 5 are unique.

14.11*. Prove that the projective planes of order 7 and 8 are unique.

14.12*. Does there exist a finite affine plane of order $q = 6$?

14.13*. Find two nonisomorphic finite affine planes of order $q = 9$.

14.14. By adding "points at infinity" to the affine geometries of orders 3, 4, 5, construct the corresponding finite projective planes.

14.15.** Give an example of a finite projective plane from which one can obtain nonisomorphic affine planes by removing one line.

14.16*. Construct a non-Desarguian projective plane of order 9.

Chapter 15

The Hierarchy of Geometries

This chapter is, in a sense, an overview of the book: in it, we try to put some order in the category of geometries by summarizing in a systematic way the relationships between the various geometries studied in the previous chapters.

We do this in the order of increasing dimension, beginning with lines (dimension one, see Section 15.1), then considering various kinds of planes (dimension two, see Section 15.2), then three-dimensional spaces (Section 15.4). (Geometries of dimension higher than three are not considered because, in my opinion, the proper place for them is in a linear algebra course.)

In Section 15.3, we give a systematic comparison of metric, affine, and projective geometries, stressing the role of distance, ratio, and cross ratio as invariants of the corresponding transformation groups.

In Section 15.5, we recall geometries with finite and discrete transformation groups and indicate their positions in the "hierarchy" of geometries described in this chapter.

The contents of this chapter (and of the whole book) are summarized in Section 15.6 and, very succintly, in Figure 15.1.

15.1. Dimension one: lines

In this section, we bring together some very simple and basic facts about the classical one-dimensional geometries. Logically, we should have presented this trivial material at the very beginning of this book, but didn't, because that would have been a very uninteresting way to start the study of geometries.

15.1.1. The Euclidean line. The Euclidean line is modeled by the real numbers $\mathbb{R}$ with transformation group $\mathrm{Ismtr}(\mathbb{R})$ consisting of parallel shifts ($x \mapsto x + v$, $v \in \mathbb{R}$) and of reflections with respect to any point. The subset of all parallel shifts is a subgroup of $\mathrm{Isom}(\mathbb{R})$, the *motion group* $\mathrm{Ismtr}^+(\mathbb{R})$ of the Euclidean line. To specify an element of $\mathrm{Ismtr}^+(\mathbb{R})$, it suffices to indicate any point of the line and its image.

The distance between two points $x, y \in \mathbb{R}$, defined in the standard way by $d(x, y) := |x - y|$, is an invariant of $\mathrm{Ismtr}(\mathbb{R})$. Given a point x and its image y, there are two elements of $\mathrm{Ismtr}(\mathbb{R})$ that take x to y: one is a parallel shift, the other a reflection in the midpoint of the segment joining x and y.

15.1.2. The hyperbolic line. The hyperbolic line can be modeled by the open interval $(-1, 1)$ with the distance function

$$d(x, y) := \frac{1}{2} |\log(\langle 1, -1, x, y\rangle)|,$$

where $\langle 1, -1, x, , y\rangle$ is the cross ratio of the points $1, -1, x, y$, i.e.,

$$\langle 1, -1, x, y\rangle = \frac{|x - 1|}{|x + 1|} \cdot \frac{|y + 1|}{|y - 1|}.$$

Isometries of the hyperbolic line include *shifts* (the one-dimensional analogs of parallel translations), given by the formula

$$T_v : [-1, 1] \to [-1, 1], \quad x \mapsto \frac{x + v}{xv + 1},$$

where v is a real number of absolute value less than 1. The composition of any two shifts is a shift, and this operation has a physical interpretation in the theory of relativity (see Section 9.4).

Shifts are not the only isometries of the hyperbolic line, there are also *symmetries in a point*; details about them are relegated to Problem 15.1.

15.1.3. The circle. The geometry of the circle $\mathbb{S}^1$ is the one-dimensional analog of spherical geometry. Its transformation group, $\mathrm{O}(2)$, consists of rotations (by angles $0 \leq \varphi < 2\pi$) and symmetries; $\mathrm{O}(2)$ contains $\mathrm{SO}(2)$, the group of all rotations, as a subgroup. To specify an element $r \in \mathrm{SO}(2)$, it suffices to indicate a single point $x \in \mathbb{S}^1$ and its image $r(x)$.

If one defines the distance between two points on the circle in the natural way (as the angle between the radii passing through them), one does *not* obtain a metric space (the triangle inequality does not hold). However, this distance function supplies the circle with a local metric space structure.

There is a natural morphism between the Euclidean line and the circle, namely the exponential covering map given by

$$\mathbb{R} \ni \varphi \mapsto e^{i\varphi} \in \mathbb{S}^1.$$

This covering plays a key role in elementary topology (e.g., for defining the degree of circle maps), and its visualization shown in Figure 16.1 in the next chapter is one of the classical icons in mathematics. The morphism exp is obviously a local (but not global) isometry.

15.1.4. The elliptic line. The elliptic line is the one-dimensional analog of Riemann's elliptic plane. It can be modeled as the circle with diametrically opposite points identified. Its transformation group $\widetilde{\mathrm{O}}(2)$ consists of rotations (by angles $0 \leq \varphi < \pi$) and symmetries (any symmetry has two fixed points whose diameters form a ninety degree angle). However, the group $\widetilde{\mathrm{O}}(2)$ is actually isomorphic to $\mathrm{O}(2)$ (rotations by φ corresponding to rotations by 2φ and symmetries acting in the same way), and it is easy to construct an isomorphism of geometries between the elliptic line and the circle. Thus the elliptic line is just another model of the geometry of the circle.

15.1.5. The affine line. For the points of the affine line $\mathbb{A}\mathrm{ff}^1$, we can take the real numbers (just as for the Euclidean line), but its transformation group, the affine group $\mathrm{Aff}(1)$, is "much bigger" than $\mathrm{Ismtr}(\mathbb{R})$; in fact, it contains $\mathrm{Isomtr}(\mathbb{R})$ as a subgroup. The group

Aff(1) consists of all transformations of the form $x \mapsto ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

To specify an affine transformation, it suffices to indicate an ordered pair of points and their images (cf. Problem 15.2).

15.1.6. The projective line. The projective line $\mathbb{R}P^1$ is obtained from the real line $\mathbb{R}$ by adding the *point at infinity* ∞ so that, topologically, it is the circle. Its transformation group consists of those bijections of $\mathbb{R} \cup \infty$ that preserve the cross ratio of any four points a, b, c, d, i.e.,

$$[(a-c)/(b-c)] : [(a-d)/(b-d)]$$

(for the definition of the cross ratio when one of the points a, b, c, d is ∞, see Problem 15.4).

Note that $\mathbb{R}P^1$ can also be defined as the set of lines passing through the origin of the plane $\mathbb{R}^2$; its transformation group is the group of nonsingular two-by-two matrices considered up to multiplication of the columns of the matrix by nonzero constants.

These two definitions of the projective line are equivalent, i.e., they determine isomorphic geometries; the proof is the object of Problem 15.5.

15.2. Dimension two: planes

Here we summarize some properties of the main objects of study in this book – the classical two-dimensional geometries. We will use the following (not very standard) terminology concerning transformation groups: we say that a transformation group possesses two *degrees of freedom* if any of its elements is determined by two points and their images. (Note that this notion does *not* coincide with the dimension of the group when the latter has a Lie group structure.)

15.2.1. The Euclidean plane. We denote the Euclidean plane by $\mathbb{R}^2$ and assume that it is familiar to the reader (For the basic properties of $\mathbb{R}^2$, the reader is referred to Chapter 0.) The plane is noncompact and orientable. Its transformation group $\mathrm{Ismtr}(\mathbb{R}^2)$ contains the subgroup of motions $\mathrm{Ismtr}^+(\mathbb{R}^2)$ which possesses two *degrees of freedom* in the sense that any motion is determined by two points

and their images. The basic invariant of this geometry is the distance between two points.

15.2.2. The sphere. We denote the sphere by $\mathbb{S}^2$. It is compact and orientable. Its transformation group, denoted by $\mathrm{Ismtr}(\mathbb{S}^2) = \mathrm{O}(3)$, contains the subgroup $\mathrm{Ismtr}^+(\mathbb{S}^2) = \mathrm{SO}(3)$ of orientation-preserving isometries, which has two degrees of freedom. The basic invariant of this geometry is the distance between two points. Note that this distance is *not* the distance induced from Euclidean space by the standard embedding of the sphere; it is the angular distance (or, which is the same thing, the geodesic distance) between points. A detailed exposition of spherical geometry appears in Chapter 6.

15.2.3. The hyperbolic plane. We denote the hyperbolic plane by $\mathbb{H}^2$. It is noncompact and orientable. Its transformation group, denoted $\mathrm{Ismtr}(\mathbb{H}^2)$, contains the subgroup $\mathrm{Ismtr}^+(\mathbb{H}^2)$ of orientation-preserving isometries (motions), which possesses two degrees of freedom. The basic invariant is the distance between two points. A detailed exposition of hyperbolic geometry appears in Chapters 7–10.

15.2.4. The elliptic plane. We denote the elliptic plane by $\mathbb{E}\mathrm{ll}^2$. It is compact and nonorientable. Its transformation group, denoted $\mathrm{Ismtr}(\mathbb{E}\mathrm{ll}^2)$, possesses two degrees of freedom. The basic invariant of this geometry is the distance between two points. A somewhat more detailed exposition of elliptic geometry appears in Section 6.7.

15.2.5. The affine plane. We denote the affine plane by $\mathbb{A}\mathrm{ff}^2$. It is noncompact and orientable. Its transformation group, denoted by $\mathbb{A}\mathrm{ff}(2)$, contains the subgroup $\mathbb{A}\mathrm{ff}^+(2)$ of orientation-preserving affine transformations, which has three degrees of freedom. The basic invariant of this geometry is the ratio of three collinear points.

15.2.6. The projective plane. We denote the projective plane by $\mathbb{R}P^2$. It is compact and nonorientable. Its transformation group, denoted by $\mathrm{Proj}(2)$, has four degrees of freedom. The basic invariant of this geometry is the cross ratio of four collinear points. A detailed exposition of projective geometry appears in Chapter 12.

15.3. From metric to affine to projective

Before passing to dimension three (in the next section), we glance at what happens when we move from two-dimensional metric geometries to the projective plane via the affine plane.

15.3.1. Metric. Many geometries (but not all) possess a natural distance function (in another terminology – a metric). This is the case for Euclidean (parabolic) geometry, as well as hyperbolic, elliptic, and spherical geometries. For such metric geometries, the corresponding transformation group (which gives them the structure of a geometry in the sense of Klein) can be defined as the group of distance-preserving transformations (isometries) with composition as the group operation.

15.3.2. Affine. Euclidean geometry in dimension two ($\mathbb{R}^2$) is a subgeometry of affine plane geometry ($\mathbb{A}\mathrm{ff}^2$): the latter is obtained from $\mathbb{R}^2$ by keeping the same set of points but increasing the transformation group. Namely, choosing any three noncollinear points O, X, Y, we fix a coordinate system in which we write affine transformations in the form

$$\begin{cases} x' = ax + by + k, \\ y' = cx + dy + l. \end{cases} \tag{15.1}$$

where $a, b, c, d, k, l \in \mathbb{R}$, $ad - bc \neq 0$, and $(x, y), (x', y')$ are the coordinates of the preimage and the image points in the basis OXY. Affine transformations either preserve or reverse orientation, according to the sign of the determinant $ad - bc$.

Unlike Euclidean geometries, hyperbolic, elliptic, and spherical geometries have no "affine counterpart", e.g., the transformation group of the hyperbolic plane cannot be increased to a larger group (like the affine group) acting on the same set of points; a more precise formulation appears in Problem 16.7.

15.3.3. Projective. The projective plane is obtained from the affine plane by adding the "line at infinity" and considerably increasing the transformation group (so that the line at infinity is not special, it is "just as good" as all the other lines, i.e., it intersects them all and can be transformed into any other line by a projective transformation).

The group Proj(2) of projective transformations can be defined via the homogeneous coordinate model or via the cross ratio of four collinear points. The group of projective transformations has an extra degree of freedom as compared to the affine group (see 15.2.6 above).

The projective plane contains, as subgeometries, not only the affine plane, but also Euclidean, hyperbolic, and elliptic geometries. It does not contain spherical geometry (see Problem 15.8).

This section is summarized by the following table.

Table: Properties of the two-dimensional geometries

	compactness	orientability	degrees of freedom	invariant
$\mathbb{R}^2$	$-$	$+$	2	distance
$\mathbb{S}^2$	$+$	$+$	2	distance
$\mathbb{H}^2$	$-$	$+$	2	distance
$\mathbb{E}\mathrm{ll}^2$	$+$	$-$	2	distance
$\mathbb{A}\mathrm{ff}^2$	$-$	$+$	3	ratio
$\mathbb{R}P^2$	$-$	$-$	4	cross ratio

15.4. Three-dimensional space geometries

In this brief section, and throughout this book, we do not study three-dimensional geometries in any detail. We have, however, mentioned and even defined three-dimensional hyperbolic, elliptic, and projective geometries ($\mathbb{H}^3$, $\mathbb{E}\mathrm{ll}^3$, and $\mathbb{R}P^3$); also, we assume that the reader is familiar with Euclidean space geometry $\mathbb{R}^3$ and, possibly, affine space geometry $\mathbb{A}\mathrm{ff}^3$.

The only goals of this brief section is to point out how the five three-dimensional geometries mentioned above are related and to connect 3D projective geometry with 2D spherical geometry, which (as mentioned above) is *not* a subgeometry of the projective plane, unlike two-dimensional elliptic, hyperbolic, and parabolic (Euclidean) geometry.

The relationship between the five 3D geometries in question is the same as that of their two-dimensional counterparts, namely: Euclidean space geometry is a subgeometry of affine space geometry,

which in turn is a subgeometry of projective space geometry; further, elliptic space geometry is a subgeometry of projective space geometry, and, finally, hyperbolic space geometry is also a subgeometry of projective space geometry.

Now the two-dimensional sphere lies in Euclidean space, and of course $\mathbb{S}^2$ is a subgeometry of the geometry $\mathbb{R}^3$, i.e., it is a subset of the space $\mathbb{R}^3$ and its transformation group $\mathrm{O}(3)$ is a subgroup of the group $\mathrm{Isom}(\mathbb{R}^3)$. By transitivity, $\mathbb{S}^2$ is a subgeometry of $\mathbb{R}P^3$.

15.5. Finite and discrete geometries

In this very brief section, we only list the finite and discrete geometries appearing in the first half of this book and indicate their relationships.

The two discrete geometries studied in Chapters 4 and 5, namely the geometries of regular tilings (Fedorov geometries) and the geometries of reflections (Coxeter geometries), are both discrete subgeometries the Euclidean plane $\mathbb{R}^2$. The finite geometries from Chapter 3, i.e., those of the regular polyhedra (Platonic geometries), are subgeometries of the geometry $(\mathbb{S}^2, \mathrm{O}(3))$; in turn, they contain many of the "toy geometries" studied in Chapter 1; other toy geometries are subgeometries of $\mathbb{R}^2$. Another series of discrete geometries is formed by the subgeometries of hyperbolic geometry briefly mentioned in Chapter 7. These geometries are, in a sense, both Fedorov and Coxeter; they are based on regular n-gons filling the hyperbolic plane. There are actually many other discrete subgeometries of hyperbolic plane geometry, but they are not studied in this book. Another class of finite geometries is constituted by the triangular Coxeter tilings of the sphere (see Section 6.6); unlike the Platonic geometries, they are subgeometries of two-dimensional spherical geometry.

Note that we do not mention the finite geometries from Chapter 14 in the present chapter because they don't "fit in" – they are not subgeometries of projective geometry.

15.6. The hierarchy of geometries

15.6.1. The hierarchical tree of geometries. This section is essentially a commentary on Figure 15.1. In it, all the main geometries

studied in this book (except the finite geometries from Chapter 14) are placed on five levels; from bottom to top, these levels are: projective, affine, metric, discrete, finite. The geometries all appear at the appropriate levels; they are joined by arrows, which stand for injective morphisms. If we regard the geometries as vertices and the arrows as edges, we obtain a directed graph; this graph is a rooted tree: starting from any vertex (geometry) the arrows lead us to the root $\mathbb{R}P^3$. This means that all the geometries we studied (except those in Chapter 14) are subgeometries of projective space geometry $\mathbb{R}P^3$.

This substantiates Cayley's famous utterance "Projective geometry is all geometry." A more correct, but less striking, formulation would be "Most geometries are parts of projective geometry."

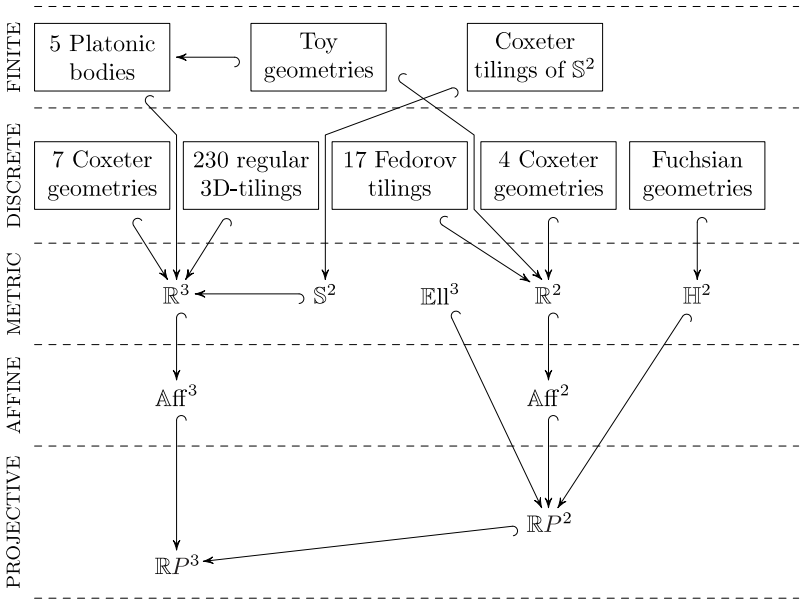


Figure 15.1. Hierarchical tree of geometries.

15.6.2. Let us climb up this "hierarchical tree of geometries". The most interesting climb begins if we first move to the vertex $\mathbb{R}P^2$ (still on the projective level); then we have the choice of three main itineraries: via affine plane geometry to the Euclidean plane (with all its beautiful discrete and finite subgeometries), or, skipping the affine level, directly to the elliptic or to the hyperbolic plane, the latter also rich in discrete geometries. (These discrete geometries, often called Fuchsian, were mentioned in Chapter 7 and in the Problems to that Chapter, but were not studied systematically in this book.)

From the root $\mathbb{R}P^3$, we could have chosen the itinerary leading us to spherical geometry and ultimately to its finite Coxeter subgeometries, or, along another limb, to the finite geometry level, reaching the Platonic geometries, which in turn contain most of the "toy geometries" – thus returning us to Chapter 1. Which of these two climbs is more beautiful, is a matter of taste.

15.7. Problems

15.1. Write out a formula allowing to find the image $S_a(x)$ of an arbitrary point $x \in (0,1)$ of the hyperbolic line under the symmetry in the point $a \in (0,1)$.

15.2. Express the parameters a and b of the affine transformation given by $x \mapsto ax + b$ in terms of the coordinates x_1, x_2 of two points and of their images y_1, y_2.

15.3. Express the parameters a, b, c, d, k, l of the affine transformation (15.1) in terms of the coordinates of three points and of their images.

15.4. Define the cross ratio of four points on the projective line when one of the points is ∞.

15.5. Show that there is a bijection between the point sets in the two definitions of the projective line given in 15.1.6 and that the corresponding transformation groups are isomorphic; prove that the two definitions yield isomorphic geometries.

15.6. Show that there is no extension of the transformation group of hyperbolic geometry and no definition of a vector that would endow the set of vectors with a vector space structure over $\mathbb{R}$.

15.7. Prove that two-dimensional spherical geometry is not a subgeometry of 2D-projective geometry.

Chapter 16

Morphisms of Geometries

In this chapter, we describe concrete examples and some classes of morphisms of geometries. They are particular cases of classes of maps normally studied in algebraic topology or Lie group courses, namely covering spaces, vector bundles, and principal G-bundles. The formal definitions of the particular cases that we consider (and call "geometric") endow the morphisms with "more structure" than that appearing in the definitions of the corresponding maps from topology courses, but, surprisingly, practically all the examples considered by topologists actually possess these structures (although topologists usually ignore them).

The chapter begins with four sections containing concrete examples of morphisms of the four types listed above. These examples include such beautiful constructions as the Hopf bundle, the Grassmannian, the Stiefel-over-Grassmann bundle, and the Milnor universal G-bundle. Their descriptions are quite elementary (although a little linear algebra and some elementary topology is needed for some of the examples), but the formal general definitions need more than the elementary prerequisites required for the previous chapters of this book. Thus, Section 4 (on Lie groups) requires the notion of

smooth manifold, a good understanding of basic linear algebra and basic topology.

The remaining sections contain the main definitions and a little theory, including two universality theorems about geometric vector bundles and geometric principal G-bundles, which yield effective constructions for obtaining all geometric vector bundles and all geometric principal bundles over a given base. Here, as in Section 4, some non-elementary mathematics is needed, but none of the standard tools of algebraic topology (the fundamental group, homotopy and homology groups) are used, the main tools being the transformation groups of the source and target geometries.

16.1. Examples of geometric covering spaces

16.1.1. Covering of the elliptic plane by the sphere. There is an obvious morphism of the geometry of the sphere $(\mathbb{S}^2 : \mathrm{SO}(3))$ onto the elliptic plane $\mathbb{E}\mathrm{ll}^2$ (see Section 6.7) obtained by identifying antipodal points of the sphere. Another way of saying this is that we consider the subgroup $\mathbb{Z}_2 \subset \mathrm{SO}(3)$ acting on the sphere by symmetry w.r.t. the sphere's center, and take the quotient of $\mathbb{S}^2$ by the two-point orbits of this action. Thus we obtain a morphism of geometries $\mathbb{S}^2 \to \mathbb{E}\mathrm{ll}^2$ such that the inverse image (which we call the *fiber*) of any point $p \in \mathbb{E}\mathrm{ll}^2$ consists of two points.

16.1.2. The exponential map. The exponential function $x \mapsto e^{ix}$, studied by the reader in calculus courses, is actually a morphism of geometries, namely the morphism $\exp : \mathbb{R} \to \mathbb{S}^1$ given by the rule $\varphi \mapsto e^{i\varphi}$, where the circle $\mathbb{S}^1$ is understood as the geometry of the set of unimodular complex numbers, i.e.,

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$$

(acting upon itself by multiplication), while the $\mathbb{R}$ is the geometry of the set of real numbers (acting upon itself by addition).

The classical picture of the exponential map (as it usually appears in elementary topology textbooks) is shown in Figure 16.1.

Note that the inverse image of any point φ of $\mathbb{R}$, called the *fiber* of the morphism exp, is in one-to-one correspondence with the integers;

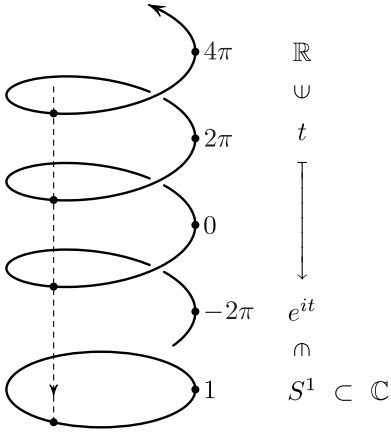


Figure 16.1. The exponential map.

in particular, the inverse image of the point $1 \in \mathbb{S}^1$ is the subgroup $\mathbb{Z} \subset \mathbb{R}$. Each fiber is also a geometry (with transformation group $\mathbb{Z}$ acting by addition).

16.1.3. The winding map $w_n : \varphi \mapsto e^{in\varphi}$. This map is a morphism of geometries $\mathbb{S}^1 \to \mathbb{S}^1$ with fiber $\mathbb{Z}_n$, the set of integers modulo n with the natural action of $\mathbb{Z}_n$ on itself by addition.

16.1.4. The partial order in winding maps of the circle. Let us say that the morphism w_n is *higher* than the morphism w_m or w_m is *lower* than w_n (denoted $w_n \succ w_m$) if there exists a morphism of geometries $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\gamma \circ w_m = w_n$ (note that here $\gamma \circ w$ means that w is performed first, then comes γ). For example, $w_6 \succ w_3$, because if we take $\gamma := w_2$, then obviously $\gamma \circ w_3 = w_6$. Clearly, $\succ$ is a partial order relation in the set of surjective morphisms onto the circle (not only in the case of the winding maps w_n, where $n = 1, 2, 3, \dots$).

There is an interesting connection between the relation $\succ$ for the morphisms w_n and the divisibility of the numbers n; see Problem 16.1.

16.1.5. The exponential map as the highest covering of the circle. The exponential map is *universal* for the winding maps of the circle in the sense that it is higher than any winding map of the circle, i.e., for any morphism $w_n : \mathbb{S}^1 \to \mathbb{S}^1$ there exists a morphism $\gamma : \mathbb{R} \to \mathbb{S}^1$ such that $w_n \circ \gamma = \exp$. The proof is relegated to Problem 16.2.

16.1.6. The flat torus. The flat torus is the unit square

$$\{(x, y) : 0 \le x \le 1, 0 \le y \le 1\} \subset \mathbb{R}^2$$

with opposite sides identified ($(x, 0) \sim (x, 1)$ and $(y, 0) \sim (y, 1)$). It is supplied with the natural metric (for example, the distance between the points $(1/2, 1 - \varepsilon)$ and $(1/2, \varepsilon)$, where $\varepsilon < 1/4$, is equal to 2ε). This metric (see Problem 16.6) gives rise to the corresponding group of isometries, which determines the structure of this geometry, that of the *flat torus*. The geometry of the flat torus is quite different from that of the torus embedded in $\mathbb{R}^3$ in the usual way, e.g., by means of the equation

$$\left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1.$$

The metric of the above embedded torus (defined via geodesics) differs from the metric of the flat torus, so that the two tori have different isometry groups and thus are different geometries.

16.1.7. Remark. Actually, neither of these two geometries on the torus is the "immanent" one. The true geometry of the torus is the one given by the *standard embedding* of the torus in complex space $\mathbb{C}^2$, i.e.,

$$\mathbb{C}^2 \supset \mathbb{T} = \left\{(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) \in \mathbb{C}^2 : r_1 = 1, r_2 = 1\right\}.$$

16.1.8. The universal covering of the flat torus. There is a natural morphism of the plane $\mathbb{R}^2$ onto the flat torus $\mathbb{T}^2_{FL}$ given by the rule

$$\mathbb{R}^2 \ni (x, y) \mapsto (x \bmod 1, y \bmod 1) \in \mathbb{T}^2_{FL},$$

which topologists call the *universal covering* of the torus. This morphism is obviously a local isometry. Its *fiber* (i.e., the inverse image of points of the torus) is the lattice $\mathbb{Z} \oplus \mathbb{Z}$. An algebraist would describe

this morphism as the quotient map taking the group $\mathbb{R} \oplus \mathbb{R}$ to its quotient by its (normal) subgroup $\mathbb{Z} \oplus \mathbb{Z}$.

16.1.9. Covering of the torus by the cylinder. There is an obvious morphism of the *flat cylinder* $\mathbb{R} \times \mathbb{S}^1$ onto the flat torus $\mathbb{T}^2_{FL}$ with fiber $\mathbb{Z}$. The details are left to the reader (see Problem 16.7).

16.1.10. Covering of the flat torus by itself. There are many geometric morphisms of the flat torus onto itself. Their investigation is left to the reader (see Problem 16.8).

16.2. Examples of geometric G-bundles

We begin this section with a description of the famous Hopf bundle, then study some other (less intricate) morphisms of geometries onto the two-sphere $\mathbb{S}^2$ (regarded as the geometry of the rotation group SO(3) acting on the sphere) with fiber (i.e., inverse image of points) the circle. The section continues with more examples of morphisms in which the main protagonists are some of the so-called "classical groups".

16.2.1. The Hopf bundle. This is one of the most intricately beautiful geometric constructions in mathematics; it has a very simple analytic description. Consider the sphere $\mathbb{S}^3$ as the subset of the two-dimensional complex space $\mathbb{C}^2$ given by the formula

$$\big\{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 + |z_2|^2 = 1\big\}.$$

We regard it as the geometry $(\mathbb{S}^3 : \mathrm{SO}(4))$.

The group $\mathbb{S}^1 = \{e^{i\varphi}\}$ acts on $\mathbb{S}^3$ by multiplication of coordinates

$$(z_1, z_2) \mapsto (e^{i\varphi} z_1, e^{i\varphi} z_2).$$

The quotient of $\mathbb{S}^3$ by this action is the complex projective line $\mathbb{C}P^1$, which is another name for the sphere $\mathbb{S}^2$ (see Problem 16.3).

Thus we obtain a surjective map $h : \mathbb{S}^3 \to \mathbb{S}^2$, called the *Hopf bundle*, whose *fiber* (i.e., the inverse image of any point of $\mathbb{S}^2$) is the circle $\mathbb{S}^1$. The orbits of the $\mathbb{S}^1$ action on the 3-sphere $\mathbb{S}^3$ are circles parametrized by the 2-sphere and filling up $\mathbb{S}^3$. These circles are

linked together pairwise, as links of a chain. An attempt to show what this looks like appears in Figure 16.2.

The figure represents $\mathbb{S}^3$ as Euclidean space with "the point at infinity" added (it lives at "both extremities" of the vertical axis of $\mathbb{R}^3$, transforming the axis into one of the orbit circles). The figure shows only two other linked orbit circles; they lie on the shaded torus; the other curves in the picture represent the sections of other concentric tori by the vertical plane of the figure.

To really visualize the Hopf bundle, the reader is referred to Etienne Ghys's home page and his "Dimensions" – a series of beautiful animations (two of them show the Hopf bundle in motion).

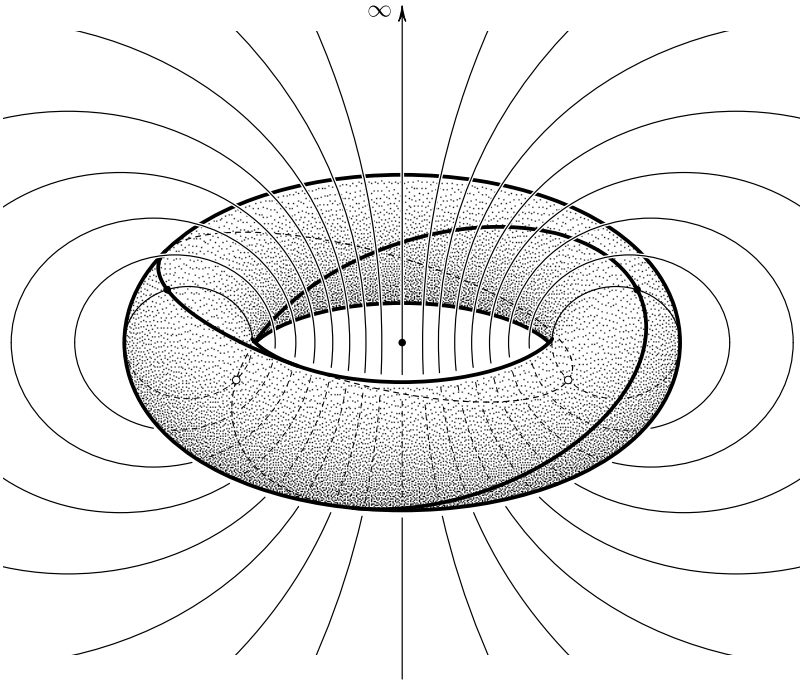


Figure 16.2. The Hopf bundle.

16.2.2. The natural morphism $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$. The Cartesian product $\mathbb{S}^2 \times \mathbb{S}^1$ possesses a natural geometry structure (Problem

16.4) so that the projection on the first factor $\mathrm{pr}_1 : (s, \varphi) \mapsto s$ is a morphism of geometries with fiber $\mathbb{S}^1$.

16.2.3. A nontrivial morphism onto $\mathbb{S}^2$ with fiber $\mathbb{S}^1$. Consider the Cartesian product $\mathbb{S}^2 \times [0, 1]$; let α be the involution of the sphere $\mathbb{S}^2$ obtained by reflection in the equatorial plane; identify all pairs of points of the form $(s, 0)$ and $(\alpha(s), 1)$. The obtained space is a three-dimensional manifold which may be supplied with the structure of a geometry (Problem 16.5). This geometry may be mapped onto the 2-sphere via the assignment

$$\mathbb{S}^2 \times [0, 1) \ni (s, t) \mapsto s \in \mathbb{S}^2,$$

and this map will be a morphism of geometries with fiber $\mathbb{S}^1$. In topology courses, the morphism in Subsection 16.2.2 is said to be a trivial $\mathbb{S}^1$-bundle (because it is the projection along the factor $\mathbb{S}^1$ of a Cartesian product), whereas the morphism from the present subsection is a nontrivial bundle.

16.3. Lie groups

This section is *not* an introduction to the (very rich) theory of Lie groups. It only contains some basic definitions and a few examples that will be used in subsequent sections. To understand the present section, the reader must know some linear algebra and be familiar with the notions of smooth manifold, smooth map, and diffeomorphism.

16.3.1. Main definitions. A *Lie group* is a smooth manifold G which is also a group such that the maps $G \times G \to G$ and $G \to G$ given by $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth. In other words, it is the geometry of a group G whose elements are the points of a smooth manifold and which acts on itself smoothly by multiplication from the right. Thus Lie groups are geometries, while *morphisms* of Lie groups are defined as morphisms of the corresponding geometries.

Suppose E and B are Lie groups, $p : E \to B$ is a surjective morphism, and $B_1 \subset B$ is a Lie subgroup; then one defines the *restriction* of p to B_1, denoted $p|_{B_1}$, in the natural way. More generally,

if $f : A \to B$ is a morphism of Lie groups, then the *pullback* of p to A, denoted f^*p, is defined by setting

$$E_1 := \big\{(a, e) \in A \times E : f(a) = p(e)\big\} \quad \text{and} \quad f^*p((a, e)) := f(a).$$

In that situation, there is a canonical morphism $E_1 \to E$ given by the formulas $\varphi(a) = f(a)$, $\Phi((a, e)) = e$.

Actually, the definition of pullback can be given in a much more general context than that of Lie groups. Thus if $p : E \to B$ is any surjective map and $f : A \to B$ is any map, then the pullback f^*p is defined exactly as above.

16.3.2. Examples of Lie groups. (1) The simplest examples of Lie groups are the groups $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$ acting upon themselves by addition.

(2) The classical groups $\mathrm{GL}(n)$ (nonsingular linear transformations of $\mathbb{R}^n$), $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ (orthogonal and orientation-preserving orthogonal transformations of $\mathbb{R}^n$), $\mathrm{U}(n)$ (Hermitian transformations of $\mathbb{C}^n$), well known from linear algebra, possess obvious Lie group structures.

(3) Such well-known groups from various parts of mathematics as the Weyl group, the group $\mathrm{SL}(n)$, the group of upper triangular matrices, all have obvious Lie group structures.

16.4. Examples of geometric vector bundles

Grassmannians (or Grassmann manifolds) G_k^n are geometries which generalize projective spaces in a natural way: their points are k-dimensional subspaces of $\mathbb{R}^n$, which in the case $k = 1$ is just the definition of $\mathbb{R}P^{n-1}$. These geometries possess a beautiful theory (involving such neat objects as Plücker coordinates, Schubert cells, etc.), but we do not go into this theory here: our aim is to describe certain morphisms of geometries in which Grassmannians play the key role.

16.4.1. Definition of Grassmann and Stiefel manifolds. The *Grassmann manifold* G_k^n is the set whose points are the k-dimensional linear subspaces L of the n-dimensional vector space over $\mathbb{R}$. This set

has a natural topological structure, a smooth manifold structure, and a geometric structure (Problem 16.9).

The *Stiefel manifold* V_k^n is the set whose points are the k-dimensional orthonormal frames of Euclidean space $\mathbb{R}^n$. This set has a natural topological structure, a smooth manifold structure, and a geometric structure (Problem 16.10).

16.4.2. Some important morphisms. (1) The *canonical Grassmann bundle* $\gamma_k^m : E_k^m \to G_k^m$, where G_k^m is the Grassmann manifold,

$$E_k^m := \big\{(L, r) \in G_k^m \times \mathbb{R}^m : r \in L\big\},$$

and γ_k^m is the natural projection $(L, r) \mapsto L$, is a morphism of geometries, provided E_k^m is supplied with the natural geometric structure (Problem 16.11). Its *fiber* $(\gamma_k^m)^{-1}(\mathrm{pt})$ is the k-dimensional vector space over $\mathbb{R}$.

(2) There are obvious inclusions $G_k^m \subset G_k^{m+1}$ and $E_k^m \subset E_k^{m+1}$, which allow us to define G_k^∞, E_k^∞, $\gamma_k^\infty : E_k^\infty \to G_k^\infty$ by passing to the inductive limit. The mapping γ_k^∞ thus obtained is a morphism of geometries that we will call the *infinite canonical Grassmann bundle*.

(3) The *Stiefel-over-Grassmann bundle* $\sigma_k^m : V_k^m \to G_k^m$, which assigns to each frame of V_k^m the linear subspace that it spans, is a morphism of geometries.

16.4.3. Universality of the canonical Grassmannian. For a certain class of geometric morphisms with fiber $\mathbb{R}^k$ (that we do not specify here, see [5], Vol. II), the infinite canonical Grassmann bundle possesses the following remarkable universality property: *for any morphism $\xi^k : E \to B$ of this class, there exists a morphism of geometries $p : B \to G_k^\infty$ such that the pullback $p^*\gamma_k^\infty$ is isomorphic to ξ^k.*

Thus the infinite canonical Grassmann bundle contains, so to speak, all the information needed to construct all morphisms from a vast class of geometric morphisms with fiber $\mathbb{R}^k$. Note that the pullback construction is an effective one, so the above theorem yields an effective method for finding new examples of morphisms of geometries.

16.5. Geometric G-bundles

In topology, a G-bundle is the quotient map from a topological space supplied with a right action of a topological group G onto the orbit space of this action. Here we study a particular case of the notion of G-bundle, in which we consider geometries rather than topological spaces and Lie groups rather than topological groups.

16.5.1. Main definitions. Let $(E:\Gamma)$ be geometry, Γ a Lie group, and G a subgroup of Γ; by a *geometric G-bundle* $p: E \to B$ we mean the projection of the space E onto the orbit space $B = E/G$ of the action of G on E.

A morphism (φ, Φ) of two geometric G-bundles $p_i : E_i \to B_i$, $i = 1, 2$, is defined as a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\varphi} & B_2, \end{array}$$

in which φ and Φ are morphisms of geometries. A morphism of geometric G-bundles is an *isomorphism* if φ and Φ are isomorphisms of geometries.

A *geometric G-bundle* $p : E \to B$ is said to be *principal* if the group G acts transitively on E. Since the action of G is transitive, each orbit is isomorphic to G; in other words, the *fiber* of the bundle p is G. The class of all principal G-bundles also forms a category.

16.5.2. Examples. (1) The identification of antipodal points on the n-dimensional sphere is a principal-$\mathbb{Z}_2$ bundle over $\mathbb{R}P^2$.

(2) The Hopf bundle (see 16.3.1) is a principal $\mathbb{S}^1$-bundle.

(3) The natural projection of the Stiefel manifold V_k^n onto the Grassmann manifold G_k^n (see 16.4.2(3)) is a principal $\mathrm{O}(k)$-bundle.

(4) The canonical Grassmann bundle $\gamma_k^m : E_k^m \to G_k^m$ is a geometric $\mathrm{GL}(k)$-bundle, but it is not principal. Its fiber is $\mathbb{R}^k$.

16.6. The Milnor construction

The Milnor construction associates with any Lie group G (in fact, any topological group) a certain geometric morphism ω_G that allows us to classify all geometric G-bundles over a given geometry. The construction is based on the notion of *join* (see, e.g., [5], Vol. I). The *join* of two topological spaces X and Y, denoted $X*Y$, is the quotient space of $X \times [0,1] \times Y$ by the equivalence relation

$$(x,0,y) \sim (x',y), \quad (x,1,y) \sim (x,1,y').$$

For example, $[0,1]*[0,1]$ is the 3-simplex, while $\mathbb{S}^1 * \mathbb{S}^1$ is the sphere $\mathbb{S}^3$. The last example is a beautiful geometric fact that every mathematician should understand, along with the (related) fact that the 3-sphere can be glued together from two "linked solid tori".

16.6.1. The construction. Let G be any Lie group; denote by

$$E_G(n) := G * G * \cdots * G \quad (n \text{ factors})$$

the n-fold iterated join of the group G with itself. Obviously,

$$G \subset E_G(2) \subset E_G(3) \subset \cdots \subset E_G(n) \subset \cdots \subset E_G,$$

where E_G is the inductive limit of $E_G(n)$ as $n \to 0$.

Consider the action of G in E_G by right shifts. The corresponding bundle

$$\omega_G : E_G \to B_G = E_G/G$$

is called the *universal geometric G-bundle*, its base is called the *classifying space* of the Lie group G. Similarly, one defines the bundle $\omega_G^n : E_G^n \to B_G^n$ called (briefly) the *n-universal G-bundle*, while its base is called the *n-classifying space* of the group G (in less shortened form, the *classifying space up to dimension n*).

16.6.2. Examples. (1) The classifying space of the group $\mathbb{S}^1$ is $\mathbb{C}P^\infty$ and the corresponding universal bundle is

$$\omega_{\mathbb{S}^1} : E_{\mathbb{S}^1} = \mathbb{S}^\infty \to \mathbb{C}P^\infty.$$

The k-classifying space of $\mathbb{S}^1$ is $\mathbb{C}P^k$ and $E_{\mathbb{S}^1}^k = \mathbb{S}^{2k+1}$.

(2) The classifying space of the group $\mathbb{Z}_2$ is $\mathbb{R}P^\infty$ and $E_{\mathbb{Z}_2} = \mathbb{S}^\infty$. The k-classifying space of $\mathbb{Z}_2$ is $\mathbb{R}P^k$ and $E_{\mathbb{S}^1}^k = \mathbb{S}^k$.

16.6.3. The universality property. For a certain class of geometric principal G-bundles (that we do not specify here, see [5], Vol. II), the universal geometric G-bundle possesses the following remarkable universality property: *for any geometric principal G-bundle $\xi_G : E \to B$ of this class, there exists a morphism of geometries $p : B \to G_k^\infty$ such that the pullback $p^*(\gamma_k^\infty)$ is isomorphic to ξ_G.*

Thus the universal geometric G-bundle contains, so to speak, all the information needed to construct all G-bundles from a vast class of geometric principal G-bundles. Since the pullback construction is an effective one, the above theorem yields an effective method for finding new examples of geometric principal G-bundles.

16.7. Problems

16.1. Given two winding maps of the circle, w_k and w_l (see 16.1.3), find the lowest winding map w_n which is higher than both w_k and w_l.

16.2. Prove the universality property of the exponential map, i.e., show that it is higher than any winding map (see 16.1.5).

16.3. Prove that the complex projective line $\mathbb{C}P^1$ is isomorphic as a geometry to the sphere $\mathbb{S}^2$.

16.4. Indicate the transformation group which supplies the Cartesian product $\mathbb{S}^1 \times \mathbb{S}^2$ with its natural geometric structure.

16.5. Indicate the transformation group which supplies the manifold constructed in 16.2.3 with its natural geometric structure.

16.6. Give a precise definition of the metric of the flat torus.

16.7. Define the geometry of the infinite flat cylinder $\mathbb{S}^1 \times \mathbb{R}$. Show that there is an uncountable infinity of different geometric morphisms with fiber $\mathbb{Z}$ of the flat cylinder onto the flat torus. How many non-isomorphic morphisms of that type are there?

16.8*. Show that the covering of the flat torus by the plane has a universality property similar to that of the exponential.

16.9. On the Grassmann manifold G_k^n, define the structure of a topological space, of a smooth manifold, and of a geometry.

16.10. On the Stiefel manifold V_k^n, define the structure of a topological space, of a smooth manifold, and of a geometry.

16.11. Define the natural geometric structure on the set E_k^m (see 16.4.2).

16.12*. Describe a map $f : \mathbb{S}^1 \to \mathbb{C}P^\infty$ such that the pullback $f^*\omega_{\mathbb{S}^1}$ is the winding map w_n.

Appendix A

Excerpts from Euclid's "Elements"

In this appendix, we present some excerpts from Book I (which deals with plane geometry) of Euclid's "Elements" (for an English translation, see [8]), interspersed by comments. I feel that some comments are absolutely necessary here, because many of the formulations appearing in the "Elements" are so worded that they sound quite weird to the contemporary reader's ear, conditioned as it is by the modern approach to mathematics. Actually, as we shall see, Euclid's wording is completely natural, once one understands the underlying meaning of geometry for the Ancient Greeks.

Let us first recall that the root "geo" stands for earth, and the root "metr" means measurement, so that the plane geometry of Euclid is an abstract version, we would say a model, of the activity of the land surveyor, working on a piece of paper (more precisely, a papyrus) rather than within the real-life landscape. Second, Euclid's contemporaries did not distinguish geometry and physics as we do; in fact, there was no science called physics at the time, and plane geometry was just the two-dimensional physical model of our universe. So Books XI–XIII of the "Elements", which treat space geometry, were nothing more for the Greeks than the theory of their (three-dimensional) physical space.

Postulates of Book I

The Postulates (we would call them axioms) are not abstract statements about points, straight lines, and circles; they simply prescribe (except Postulate IV) what the geometer (i.e., land surveyor) can draw on his papyrus. Euclid writes:

Let the following be postulated:

I. *To draw a straight line from any point to any point.*

II. *To produce a straight line continuously in a straight line.*

III. *To describe a circle with any center and distance.*

IV. *That all right angles are equal to one another.*

V. *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Of course, our land surveyor cannot draw an infinite line, and the expression "straight line" in Postulates I, II, V means "line segment" in our terminology, and this is what makes Postulate II necessary. (Note that this postulate cannot be adequately translated into a meaningful statement in the modern terminology.)

Note, further, that no uniqueness statements appear in Postulates I, II, III. This is quite natural, because all three are prescriptions for admissible well-defined actions of the land surveyor, and it goes without saying that, for Euclid's contemporaries, there was only one way to perform this action.

Postulate III sounds unnecessary to our ear; we would simply *define* the circle as the locus of all points lying at the given distance (the radius) from a given point (the center of the circle). But it is, for the Greeks, an informative assertion, saying, as it does, that drawing a circle is an admissible and well-defined action. We must also remember that for the Greeks the circle, like the line, is not a set of points, but a certain geometric entity.

Postulate IV is different from the other four in that it is an observation rather than a guide to action. In it, the word "equal" is

used. What does it mean? Should it be understood as "congruent"? We shall return to this question below.

Finally, a few comments on the Fifth Postulate (once described as "the most important utterance in the history of science"). What we regard as a better formulation of this axiom, namely *one and only one parallel to a given line passes through a given point*, is absolutely inadmissible for Euclid. Indeed, it involves the notion of infinite line (which the geometer cannot draw) and is, essentially, a negative statement, saying that there *does not exist* a common point to two specific (infinite!) straight lines. On the contrary, Euclid's version is very constructive: it says that if a geometer performs certain actions, he will obtain the intersection point of two lines, and even indicates the whereabouts of this point.

The Common Notions

For us Euclid's "common notions" sound like axioms specifying the meaning of the word "equal", but what did Euclid have in mind? This is what he says:

1. *Things which are equal to the same thing are also equal to one another.*

2. *If equals be added to equals, the wholes are equal.*

3. *If equals be subtracted from equals, the remainders are equal.*

4. *Things which coincide with one another are equal to one another.*

5. *The whole is greater than the part.*

From the modern point of view, the first and fourth common notions are the transitivity and the reflexivity of the equality relation, and (if one adds symmetry, which for the Greeks was undoubtedly implicitly assumed), we get the definition of an equivalence relation in the set-theoretical sense.

But what are the "things" that appear (explicitly and implicitly) in all five of the above statements? Are they geometric entities or are they numbers (e.g. lengths of segments) as one is led to suspect

upon reading the second and third common notions? Actually, they are both: we must remember that the Ancient Greeks had no theory of real numbers, the (lengths of) line segments *were* the real numbers. Thus, when the Pythagoreans discovered the irrationality of $\sqrt{2}$, this fact was stated as the incommensurability of two line segments: the diagonal of the unit square and its side. Further, "things" are not only line segments, they can also be (the measure of) an angle, the area of something, etc.

Finally, what does the word "equal" mean? Congruent? What does the word "greater" in the fifth common notion mean? These are difficult questions, but one should have in mind that nowhere does Euclid mention any transformations of the plane, there are no parallel translations, no rotations, no superpositions of triangles in Euclid's Book I, so that there are no hints whatsoever in it about the Klein approach to geometry based on transformations. Thus equal triangles are those which have three equal sides and whose corresponding angles are also equal (however, a kind of superposition argument does appear in the proof). It is less clear what the words "added" and "subtracted" in **1** and **2** mean, but judging from some of the proofs in Book I, they refer to the union and set difference of geometric objects, although of course such entities as angles and line segments are not defined as sets of points.

The Definitions of Book I

We mentioned in Chapter 11 that the Ancient Greeks realized that to develop geometry as a deductive science based on rigorous proofs without any logical vicious circles, one had to start by formulating certain statements without proof – the *postulates* (axioms in our terminology). However, they did not use the same type of argument for definitions, and so did not believe, as we do, that in a rigorous mathematical theory you cannot define concepts in terms of each other without vicious circles, unless you start with some basic *undefined notions* which are only named and not specified in any way.

But we should not regard the absence of undefined concepts in Euclid as a logical defect of his exposition. Indeed, the geometric

entities considered were, for Euclid, objects of the physical world and could be meaningfully described as such. Let us see how he describes them.

1. *The point is that which has no part.*

2. *A line is breadthless length.*

3. *The extremities of a line are points.*

4. *A straight line is a line which lies evenly with the points on itself.*

5. *A surface is that which has length and breadth only.*

6. *The extremities of a surface are lines.*

7. *A plane surface is a surface which lies evenly with the straight lines on itself.*

I have always been struck by the beauty and the depth of these descriptions. Perhaps the most striking fact is that the first concepts defined (except the straight line in Definition 4 and the plane in Definition 7) are basic *purely topological* notions, namely points, lines (we would say curves), surfaces, and extremities (we would say boundary operators).

The notion of point (defined as an irreducible entity) is close to physics (points are defined similarly to atoms). The description of straight lines is beautifully mysterious. Is it a poetic surrogate of the idea of geodesic? Or of translational invariance along itself?

Euclid continues with a description of angles formed by curves and by straight lines, which in the English translation is rather poetic because of the double meaning of the word "inclination":

8. *A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

9. *And when the lines containing the angle are straight, the angle is called rectilineal.*

Here let us note first that Euclid begins with the most general case of an angle between curves (rather than straight lines). It should also be noted that the definition of *rectilineal angle* in Definition 9 is

the *first* characterization of a basic notion by Euclid that we can regard as a mathematical definition in the modern sense, as opposed to the eight previous ones, which are merely intuitive descriptions. Note also that what an angle is (say a rectilineal one), actually, is not clear from the definition: is it a pair of straight lines, or a pair of rays, or part of the plane bounded by them?

The next three definitions (which also concern angles) can also be regarded as mathematical in the modern sense, provided that the use of the common notions "equal", "greater", "less" are allowed.

10. *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.*

11. *An obtuse angle is an angle greater than a right angle.*

12. *An acute angle is an angle less than a right angle.*

Note that Definition 10 actually defines two different (although closely related) concepts: right angles and perpendiculars.

The next two definitions are remarkable for their topological generality.

13. *A boundary is that which is an extremity of anything.*

14. *A figure is that which is contained by any boundary or boundaries.*

The word "extremity" previously appears in Definitions 3 and 6, and apparently was easier to grasp for Euclid's contemporaries that the synonymous word "boundary", hence Euclid uses it to give a descriptive definition of the notion of boundary in Definition 13. Note also that it is implicit in Definition 14 that a boundary is (in our terminology) a connected set. In that definition, the use of the expression "contained by" rather than "contained in" is crucial; we would say "bounded by" in this context, but that expression can hardly be used in the formulation of Definition 14, which would then obviously sound like a tautology.

The next four definitions concern circles. More precisely, what Euclid calls a circle would be called a disk by the modern mathematician; today the word circle stands for the boundary of the disk.

15. *The circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.*

16. *And that point is called the center of the circle.*

17. *A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line bisects the circle.*

18. *A semicircle is the figure contained by the diameter and the circumference cut off by it. The center of the semicircle is the same as that of the circle.*

Of course one should keep in mind that, as before, "straight line" means "line segment" (in modern terminology) and for "semicircle" we would say "half-disk".

The next four definitions are about various polygons, including triangles, squares, rectangles (called "oblongs" by Euclid), and other types of quadrilaterals.

19. *Rectilineal figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.*

20. *Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides equal, and a scalene triangle that which has its three sides unequal.*

21. *Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.*

22. *Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not*

equilateral; a rhombus that which is equilateral but not right-angled; a rhomboid that which has its opposite sides and angles equal to one another, but is neither equilateral nor right angled. And let quadrilaterals other than these be called trapezia.

Definitions 19–22 are organized with esthetic flair, their symmetric repetition is almost poetic, and pleases even the contemporary ear. The reader has surely noted the use of some unusual terms: "rectilineal figure" for "polygon", "rhomboid" for "parallelogram" (more precisely, a rhomboid is a generic parallelogram, i.e., one which is neither a rhombus nor a rectangle). It is interesting that all the objects defined in 19–22 are generic: in contrast with modern elementary geometric terminology, for Euclid the square is not a particular case of the rectangle (oblong), the rectangle is not a particular case of the rhombus, etc.

Euclid concludes his list of definitions with the crucial definition of parallel lines.

23. *Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet each other in any direction.*

Notice that Definition 23 is a perfectly rigorous mathematical definition, but it does *not* assert the *existence* of parallel lines.

Why has Euclid postponed this definition to the very end of his list? Clearly, it does not use any terms appearing in Definitions 7–22, so why didn't he place it earlier, say right after Definition 7? I believe the reason for that is his dislike of the Fifth Postulate, which he tries to avoid using as long as he can. Thus the word "parallels" first appears in the formulations of the propositions of Book I only in Proposition 27, and the construction (existence) of parallels is not asserted until Proposition 31.

The Propositions of Book I

Here we state the theorems (propositions) of Euclid's treatment of plane geometry in the order of their appearance, without the proofs and figures and with a minimum of comments. The key point here

is the order in which the propositions are proved. The reader should note that many of the propositions assert the possibility of performing a certain specific geometric construction (so that they can be understood as existence theorems); however, it can be argued that in many cases, when the possibility of a construction is claimed, it is tacitly assumed that the construction is well defined, which means that often such propositions were understood by the Ancient Greeks as existence *and uniqueness* theorems.

1. *On a given straight line to construct an equilateral triangle.*

Of course the construction was carried out by means of a compass (so that Postulate 3 was used twice), and the fact the two constructed circles intersect was considered obvious (which indeed it is).

2. *To place at a given point* (*as an extremity*) *a straight line equal to a given straight line.*

3. *Given two unequal straight lines, to cut off from the greater a straight line equal to the lesser.*

Of course in these three statements, and in the subsequent ones, the words "straight line" mean "line segment" (in our terminology).

4. *If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*

This "first test of congruence of triangles" is familiar to high school students all over the world and denoted by SAS in North America. Note that there is no mention of congruence or of some kind of motion or superposition in the proposition. I don't think that Euclid, saying "the triangle will be equal to the triangle" implied some sort of congruence, and the end of the proposition simply explains what is meant by equal triangles, namely the equality of all the corresponding elements (sides and angles) of the triangles.

5. *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

The reader will have noted the use of the word "base" in the last two propositions and the word "under" in the last one, which of course means that Euclid accompanied the propositions with pictures, in which the "bases" of triangles were drawn as horizontal lines.

6. *If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.*

Proposition 6 is the reciprocal of Proposition 5. In many high school geometry courses, the two statements are united in an if and only if theorem.

7. *Given two straight lines constructed on a straight line* (*from its extremities*) *and meeting in a point, there cannot be constructed on the same straight line* (*from its extremities*)*, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.*

Now this is a statement leading up to the "third test of congruence of triangles" (SSS), and two cases must be considered in its proof because of possible orientation reversal. Remarkably, Euclid first states this as a uniqueness theorem, using the expression "on the same side of it" to get rid of the nonuniqueness. Note that the expression in quotes first appears in another context, namely in the statement of the Fifth Postulate.

8. *If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.*

The reader should have recognized the conditions as the assumptions of the "third test of congruence of triangles" (SSS) taught in high school geometry courses. However, the assertion of this proposition is weaker (than the one in the "test"); it only claims the equality of one angle rather than three. The appearance of the term "base" is rather curious – it is not a rigorous mathematical term, it simply shows that Euclid and his followers would draw one of the sides of a triangle horizontally, and that side would be called the base of the triangle.

9. *To bisect a given rectilineal angle.*

10. *To bisect a given finite straight line.*

11. *To draw a straight line at right angles to a given straight line from a point given on it.*

The three theorems 9–11 are all proved by simple compass and straight edge constructions.

12. *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.*

This key assertion (with the explicit stipulation of uniqueness added) is often taken to be one of the axioms in 20th century axiomatic expositions of plane geometry. Note the appearance of the expression "infinite straight line", which used here instead of "straight line" (in the sense of line segment), because otherwise the statement would be false.

13. *If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.*

Of course, by "angles equal to two right angles", Euclid means angles whose geometric sum equals two right angles.

14. *If with any straight line, and at a point of it, two straight lines not lying on the same side make adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.*

This theorem may be difficult to understand for the reader. In modern terminology, it says that if two line segments with an extremity at the same point A of a line l form angles with the line l whose sum is $180°$, then the two segments lie in the same straight line.

15. *If two straight lines cut one another, they make the vertical angles equal to one another.*

16. *In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.*

In the English translation of Euclid's text used here, the logical meaning of the construction "either of ... and" may not be clear, but

of course Euclid and his followers understood the statement of this theorem correctly.

17. *In any triangle two angles taken together in any manner are less than two right angles.*

Note that this theorem shows that we are not in elliptic geometry, but it does not contradict hyperbolic geometry (Euclid has not used the Fifth Postulate in any of the proofs so far).

18. *In any triangle the greater side subtends the greater angle.*

19. *In any triangle the greater angle is subtended by the greater side.*

20. *In any triangle two sides taken together in any manner are greater than the remaining one.*

In its modern formulation, this fundamental statement is known as the "triangle inequality", the key axiom in the definition of metric space.

21. *If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.*

22. *Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.*

This sounds like an existence theorem for triangles (related to the SSS test), but it also contains a clear formulation of the triangle inequality.

23. *From a given straight line and a point on it to construct a right angle equal to a given right angle.*

24. *If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.*

25. *If two triangles have the two sides equal to two sides, respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.*

Concerning the term "base", see the remark after Proposition 8.

26. *If two triangles have the two angles equal to two angles and one side equal to one side, namely, either the side adjoining the equal angles, or the side subtending one of the angles, they will also have the remaining sides equal to the remaining sides and the remaining angle equal to the remaining angle.*

This is the "second test of congruence of triangles" (ASA) in its full generality, and it is used in the proof of the next proposition.

27. *If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.*

It is only at this point that Euclid makes use of the Fifth Postulate, and it is in this proposition that the word parallels first appears (after the definitions).

28. *If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.*

29. *If a straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.*

30. *Straight lines parallel to the same straight line are also parallel to one another.*

31. *Through a given point to draw a straight line parallel to a given straight line.*

Here Euclid resorts (after a long interlude) to a statement asserting that our "land surveyor" can perform a certain construction. Note that this proposition is close to the formulation of the "axiom

of parallels" as it usually appears in high school geometry courses. Euclid does not explicitly state that the construction is unique; however, as we explained above, the uniqueness of the construction was usually implicit in statements of this type.

32. *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

This is one of the key theorems of Euclidean geometry (which differentiates it from the elliptic and hyperbolic geometries). It is proved by using Proposition 31 to draw a parallel to the base through the opposite vertex and applying Proposition 29 to compare the angles formed at that vertex with the interior angles at the base.

33. *The straight lines joining equal and parallel straight lines* (*at the extremities which are*) *in the same directions* (*respectively*) *are themselves also equal and parallel.*

34–45. These twelve propositions have to do with constructions of parallelograms and triangles which involve parallels. When speaking of the equality of triangles and parallelograms in these propositions, Euclid means equality of areas (and not what we would call isometry or superposition by a motion). Thus the phrase (from Proposition 41) "the parallelogram is double the triangle" means that the area of a certain parallelogram is twice that of a certain triangle.

46. *On a given straight line describe a square.*

Recall that the very first proposition asserted the possibility of constructing the "perfect triangle" (an equilateral one) and now, almost at the end of Book I, Euclid shows that it is possible to construct the "perfect quadrilateral", i.e., the square. This search for perfection is characteristic not only of Greek mathematics (recall the Platonic bodies), but of Greek art and culture in general.

47. *In right angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

This is the famous Pythagorean theorem. Note that it is a purely geometric statement, it does *not* say that $a^2 + b^2 = c^2$, it asserts that

the area of the square constructed on the hypotenuse is the sum of areas of the two squares constructed on the other two sides. Euclid's proof is not the familiar one (called "Pythagoras' pants" in some countries and obtained by cutting the two squares constructed on the shorter sides of the right triangle into triangles that fit together to form the square constructed on the hypotenuse) – he cuts the square constructed on the hypotenuse into two rectangles by the prolongation of the altitude issuing from the right angle and shows that each of these rectangles has the same area as the corresponding square.

48. *If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.*

Thus Euclid concludes Book I with the reciprocal statement to the Pythagoras theorem.

Conclusion

It is customary among mathematicians to look down on Euclid and criticize the lack of rigor of his development of geometry. To my mind, this attitude only demonstrates the narrow-mindedness of such critics, their absolutization of what was regarded as rigor in the 19th century. Here I have tried to show that Euclid "Elements" have their own internal logic, and, unlike modern axiomatic theories, possess a striking beauty which makes Euclid's book one of the greatest achievements of human culture.

Appendix B

Hilbert's Axioms for Plane Geometry

In this appendix, we present Hilbert's axioms for plane geometry, which he first developed in a series of lectures at the University of Göttingen in 1898–1899. Following Hilbert (see the English translation of his celebrated *Gründlagen* in [9]), we then prove the consistency of his theory (i.e., we show that there are no contradictions in it, provided that there are none in the theory of algebraic numbers).

Hilbert's axioms constitute the first rigorous (in our present understanding of the word) treatment of plane geometry in axiomatic form. When Hilbert's axiomatics was published, it was common knowledge that a rigorous construction of plane geometry is possible within the framework of the theory of real numbers by using Cartesian coordinates, thus transforming plane geometry into a particular case of linear algebra over $\mathbb{R}$. Moreover, by then Hermann Weyl showed that, in the same framework, one could develop geometry in coordinate-free form. Nevertheless, Hilbert's work was a fundamental breakthrough in the understanding of geometry, and remains an important milestone in the history of mathematics.

The main distinctive feature of Hilbert's approach is that he understood and implemented the idea that in a rigorous exposition of a mathematical theory *some basic concepts and relations must be left*

undefined, for otherwise a logical vicious circle necessarily appears in the definitions. In his exposition, the undefined notions are "point", "straight line", "plane", "belongs to", "lies between", and "is congruent to". Thus the axioms give, in a sense, an implicit definition of these undefined notions.

In his book [9], Hilbert not only lists the axioms, but also indicates the basic facts of plane geometry in the order in which they can be derived from the axioms. Also, in Hilbert's exposition, the axioms related to space geometry are not separated from those of plane geometry; so in our exposition we simply omit the axioms (or the parts of the axioms) dealing with space geometry (and modify the numbering of the axioms accordingly).

In this appendix, as in the previous one, we look at and comment on the axiomatics of plane geometry from the point of view of a historian. As explained in the Preface, this is due to one of the author's biases: I believe that Euclidean plane geometry should not be taught by means of axiom systems such as Euclid's or Hilbert's, its object is a mathematical entity whose main protagonist is the real line, and no rigorous study of plane geometry is possible unless we know what the real numbers are. For that reason, although we reproduce the wording of the axioms verbatim (in their English translation appearing in the book [9]), we replace Hilbert's mathematical commentary concerning the axioms and their consequences by some remarks of historical nature.

I. Axioms of connection

This first group of axioms establishes a relationship between the undefined concepts *point* and *straight line* expressed by means of the (also undefined!) relation *belongs to*. (For the sake of brevity, in what follows we often write "line" instead of "straight line".) It is very important to understand that in Hilbert's exposition a line is a line is a line (an undefined concept) – it is *not* a set of points as we have been taught!

As to the relation "belongs to", it has many synonymic versions: instead of saying "the point A belongs to the straight line l" we can say "l passes through A", "l contains A", "A is a point of l", etc. If A

belongs to a line l and also belongs to another line m, we say that "A is the common point of lines l and m", or "the lines l and m intersect at the point A", etc. If two points A and B belong to the line l, we can also say that "l joins A and B", "A and B determine l", etc.

I, 1. *Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.*

Hilbert explains what "A and B determine a" means in his commentary, but what is meant by "completely determined" is not explained. Should it be understood as stipulating the uniqueness of the line determined by A and B? Apparently not, because the next axiom is a uniqueness axiom of sorts. If one wishes to be very formalistic, it should be noted that at this stage we don't know that two distinct points actually exist. But it is reasonable to suppose that the non-emptiness of the set of points and of the set of lines is tacitly assumed by Hilbert.

I, 2. *Any two distinct points of a straight line completely determine that straight line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.*

Note that at this stage we do not know that there are three distinct points A, B, C on any line, in fact, we do not even know that there are two. This assumption appears later, in the seventh of Hilbert's axioms of connection. We reproduce only the part of the latter axiom that concerns plane geometry and change its number from seven to three. (Hilbert's axioms of connection 3, 4, 5, 6 have no bearing on plane geometry, they are about planes in space.)

I, 3. *Upon any straight line there exist at least two points, and there are three points not belonging to any straight line.*

One of the main consequences of these axioms is that two distinct lines have no more than one common point.

The axioms of connection are satisfied by the finite geometry consisting of three "straight lines" each consisting of two points and such that any two of the lines have one point in common.

II. Axioms of order

This group of axioms involves, besides the previously mentioned undefined notions (point, line) and the relation "belongs to", a new undefined relation *lies between*. Hilbert explains in his commentary that this relation is a relation of order and even presents a picture of point B lying between points A and C on a straight line as an illustration for the first axiom of order, which reads:

II, 1. *If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.*

II, 2. *If A and C are two points of a straight line, then there exists at least one point B between A and C and at least one point D so situated that C lies between D and B.*

Note that the second axiom of order implies that there is an infinite (at least countable) number of points on each line.

II, 3. *Of any three points of a straight line, there is always one and only one which lies between the other two.*

II, 4. *Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.*

The rather awkward formulation of the fourth axiom (it doesn't sound any better in German) using the strange expression "so arranged" really means that any four points on a line may be denoted by A, B, C, D in such a way that B lie between A and C, etc.

The first four axioms of order allow Hilbert to define the notion of *segment* AB as the set of points lying between A and B; the points A and B themselves are called the *extremities* of the segment AB. Note that the extremities do not belong to the segment, so that we would call AB an open interval.

The notion of order also allows us to define in the natural way what Hilbert calls a *half-ray* (we would use the term ray instead): a half-ray originating from the point A and passing through the point

B different from A is the set of all points lying on the line AB on the same side of A as B.

II, 5. *Let A, B, C be three points not lying in the same straight line and let a be a straight line not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment BC, then it will also pass through either a point of the segment AC or of the segment AB.*

This statement is known as *Pasch's axiom*. As worded by Hilbert, it is incorrect, because it can happen that the line a passes through the point A, which is neither a point of the "segment" AC nor of the "segment" AB.

The axioms of order allow Hilbert to define the notions of half-plane, although he never uses such a set-theoretical term, referring to points lying *on the same side* (or *on different sides*) of a line in the plane or *on the same side* (or *on different sides*) of a point on a line. The reader will recall that in Euclid's *Elements* the expression "on the same side" was never defined and was apparently regarded as obvious, in particular, in the Fifth Postulate.

Further, Hilbert defines the notions of *broken line* and of *polygon*, and as particular cases of the latter, *triangles, quadrangles, pentagons,..., n-gons*. He then states the Jordan Curve Theorem for polygons, and claims that it can be obtained "without serious difficulty."

III. Axiom of parallels

Hilbert's version of Euclid's Fifth Postulate reads:

III, 1. *In the plane there can be drawn through any point A lying outside of a straight line a one and only one straight line which does not intersect the line a. This straight line is called the parallel to the line a through the given point A.*

It is interesting to note that in this axiom Hilbert resorts to traditional terminology when he writes "there can be drawn" rather than something more formal like "there exists". Also note that for Hilbert the relation of parallelism is not an equivalence relation (it is not

reflexive) and its definition is included in the formulation of the axiom. Unlike Euclid, Hilbert does not separate the definitions and the axioms, the definition of parallelism appearing in Axiom III, 1.

IV. Axioms of congruence

In this group of axioms, another undefined concept appears, that of *congruence*. In modern expositions of plane geometry, it is usually explicitly defined: two figures are congruent if there exists an isometry (i.e., a distance-preserving transformation of the plane onto itself) that takes one of the figures to the other. This approach is of course unacceptable to Hilbert, because it is based on the notion of *distance*, which never appears in Hilbert's axiomatization. In Hilbert's geometry, as well as in Euclid's, there is no fixed unit of measure.

IV, 1. *If A and B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment $A'B'$. We indicate this relation by writing*

$$AB \equiv A'B'.$$

Every segment is congruent to itself; that is, we always have

$$AB \equiv AB.$$

This axiom may be described briefly as saying that any segment can be *laid off* upon a given side of a given straight line in one and only one way. Note that the straight line a is not really needed in the statement of the axiom. Note also the expression "we can always find", used here instead of the more formal "there exists", and the fact that congruence (at least for segments) is a reflexive relation (which is explicitly specified in the axiom); however, it is not specified that this relation is symmetric and transitive. The symmetry of the congruence relation will be proved later. As to transitivity, it is explicitly required in the next axiom.

IV, 2. *If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to*

the segment $A''B''$; that is, if $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

The reader will readily show that Axiom IV, 3 (together with IV, 1) implies the reflexivity of the congruence relation.

The next axiom says that adding congruent segments end to end one gets congruent segments. It reads:

IV, 3. *Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or another straight line a' having, likewise, no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$, we have $AC \equiv A'C'$.*

The first three axioms of congruence allow us to give several rigorous definitions related to the notion of angle. That of *angle* as two half-rays (called the *sides* of the angle) drawn from the same point (called the *vertex* of the angle) and lying on two different straight lines, and of the *interior of an angle* as the set all points of the plane not lying on the sides of the angle and such that any two points of the set can be joined by a segment not intersecting the sides.

In the next axiom the (undefined) congruence relation is applied to angles; the axiom says that one can "lay off" an angle congruent to a given one upon a given side of a half-ray. In Hilbert's formulation:

IV, 4. *Let an angle (h, k) and a line a be given in the plane. Suppose also that a definite side of the line a is given. Denote by h' a half-ray of the line a and emanating from a point O on that line. Then there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a. We express this relation by means of the notation*

$$\angle(h, k) \equiv \angle(h', k'),$$

Every angle is congruent to itself; that is,

$$\angle(h, k) \equiv \angle(h, k)$$

or

$$\angle(h, k) \equiv \angle(k, h).$$

Note that Hilbert takes great pains to explicitly state that the equality of angles is reflexive and stipulates that an angle does not depend (up to congruence) on the order in which its sides are indicated. Just as for segments, the symmetry of the congruence relation for angles is not explicitly stipulated, but is proved later.

The final axiom of this group has to do with two triangles; it is similar to a theorem known as the "first test of congruence of triangles" in the high school geometry textbooks of many countries, but its conclusion does *not* claim that the two triangles are congruent, for the excellent reason that the notion of congruence of triangles is an undefined one, and that its definition will be given later.

IV, 5. *If, in two triangles ABC and $A'B'C'$, the congruences*

$$AB \equiv A'B',\ AC \equiv A'C',\ \angle BAC \equiv \angle B'A'C'$$

hold, then the congruences

$$\angle ABC \equiv \angle A'B'C' \text{ and } \angle ACB \equiv \angle A'C'B'$$

also hold.

The reader will have noticed that this is the first triangle congruence test (SAS).

Once all the axioms of congruence have been stated, Hilbert gives a few more definitions and some important consequences of all the axioms, but at first without using the axiom of parallels. The definitions include the notions of *supplementary* and *vertical* angles and of a *right angle* (the latter is defined as an angle congruent to its supplementary angle). This is followed by an *explicit definition* of *congruent triangles* as those having all their sides and all corresponding angles congruent.

After this Hilbert proves the *First, Second*, and *Third Theorem of Congruence of Triangles*. This is followed by two theorems on the congruence of angles (in particular, the theorem asserting that if two angles are congruent, then so are their supplementary angles). These theorems are used to prove that any two right angles are congruent. At this point Hilbert points out that Euclid held this last fact as one of the postulates, "although it seems to me wrongly", he comments.

Then, using the word "figure" to mean a finite set of points, he states what he calls "the most general theorem relating to congruence". It asserts that if $(A, B, C, \dots)$ and $(A', B', C', \dots)$ are congruent figures and P is any point, then it is always possible to find a point P' such that the figures $(A, B, C, \dots, P)$ and $(A', B', C', \dots, P')$ are congruent; furthermore, if the two figures have three points not lying in a straight line, then the point P' is unique.

At this point Hilbert finally makes use of his Axiom of Parallels and states the theorem on the congruence of the appropriate (alternate-interior and exterior-interior) angles obtained by cutting two parallel lines by a third line. It is interesting to note that Hilbert ignores considering the logical possibility of the third line cutting one of the two parallels but not the other. The fact that this possibility cannot actually occur is an obvious consequence of Hilbert's Axiom of Parallels, and I suppose Hilbert felt that this would be obvious to any reader.

The last theorem that Hilbert states after the axioms of the first four groups reads: *The sum of angles of a triangle is two right angles.* He does not explain what is meant by the "sum" of angles, but of course the sum is understood in the geometric sense: the theorem does *not* mean that the sum (of measures) of the three angles is 180°, it means that if the angles are laid off from some straight line in succession, then the second half-ray of the third angle will lie on the straight line from which we began laying off the angles of the given triangle.

It is of course no accident that Hilbert stops at that point: undoubtly, he understands the crucial role of this theorem in the context of Euclidean and non-Euclidean geometries.

V. Axiom of continuity

This is the last axiom in Hilbert's axiomatics. It is commonly known as the *axiom of Archimedes* and ordinarily appears in the axiomatic definition of the real numbers. In its geometric form, it says that by laying off any segment along a line a sufficient number of times we will eventually reach any given point on the line. Hilbert words it as follows.

V. *Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points $A_2, A_3, A_4, \ldots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4, etc. Moreover, let the segments*

$$AA_1,\ A_1A_2,\ A_2A_3,\ A_3A_4, \ldots$$

be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.

The reader may have noticed the word "equal" in the formulation of the axiom. This is a significant slip of the tongue – Hilbert means "congruent" here, since he is dealing with segments, and not lengths of segments (real numbers).

Although in Hilbert's first publication of his axiomatics the above axiom is the last one, in the French translation of his book he added one more axiom, which he called the *axiom of completeness*. This axiom is, as Hilbert says, "not of purely geometric nature" and its *raison d'être* is to ensure that the points of any straight line be in one-to-one correspondence with the real numbers, thus making Hilbert's axiomatics categorical. We postpone our discussion of these questions to the end of the next subsection.

Consistency of Hilbert's axioms

Hilbert establishes the *consistency* of his axiomatics (i.e., shows that his axioms are noncontradictory) by constructing a *model* of his plane geometry.

To do this Hilbert denotes by Ω the set of algebraic numbers obtained from the number 1 by the four arithmetical operations and the operation $\sqrt{1+\omega^2}$, where ω is one of the previously defined numbers. The set Ω consists of real numbers and is obviously countably infinite. In the model, a pair (x, y), where $x, y \in \Omega$, is called a *point*, a triple $(u : v : w)$, where $u, v, w \in \Omega$ and $u^2 + v^2 \neq 0$, is called a *straight line*, and we say that the point (x, y) *belongs to* the line $(u : v : w)$ if

$$ux + vy + w = 0.$$

Then Hilbert notes that Axioms I and III are obviously fulfilled. Further, using the usual order relation $(x < v)$ for the numbers in Ω,

he defines the relation *lies between* for three points on a line in the natural way. It is easy to verify that then the axioms of group II also hold.

Defining the notion of *congruence* as it is usually done in analytic geometry, Hilbert points out that the axioms of group IV (including those about laying off segments and angles) are fulfilled as well. This leaves the only axiom of group V (that of Archimedes), which of course holds for the algebraic numbers $\Omega \subset \mathbb{R}$.

Thus Hilbert has constructed a model of Euclidean plane geometry in the arithmetic related to the algebraic numbers; in this model, the undefined terms "point", "straight line", "belong to", "lie between", "be congruent" acquire a concrete interpretation so that all the axioms of Euclidean plane geometry are fulfilled.

This means that *any contradiction resulting from Hilbert's system of axioms must also appear in the arithmetic related to* Ω. Thus we can say that *Hilbert's axiom system is consistent* provided that there are no contradictions in the theory of algebraic numbers.

Conclusion

It should be noted, however, that in Hilbert's model of plane geometry *the number of points, as well as the number of lines, is countable.* Of course it is possible to construct a noncountable model of this geometry, with the set of points having the cardinality of the continuum, simply as it is done in analytic geometry courses based on Cartesian coordinates. Therefore, Hilbert's axiomatics is, as logicians say, *noncategorical*, which means that the axiomatics can have nonisomorphic models.

Regarding this as a drawback, Hilbert added, in later editions of his work, a *Completeness Axiom* which asserts, roughly speaking, that no extra points may be added to any line without contradicting the other axioms. This axiom, together with the axiom of Archimedes, readily implies that there is an order-preserving correspondence between the set of points belonging to a straight line and the real numbers.

Why didn't Hilbert make things much easier for himself and his readers by coming right out and stating this simple and fundamental fact as an axiom? For one thing, he undoubtedly wanted his geometry to be formally independent of the theory of real numbers, being a kind of "pure geometry". However, there is no way of going around the fact that any rigorous treatment of what we regard as Euclidean plane geometry will result in implicitly constructing the theory of real numbers as the set of all points on a line, addition being defined by putting segments end-to-end, order by the betweenness relation, and multiplication via homothety. I tend to believe that Hilbert purposely went to great pains to hide the fundamental role of the field $\mathbb{R}$ in his constructions, because he was aware that Euclidean geometry could be rigorously constructed in a much simpler way by using $\mathbb{R}$ from the very beginning, rather than as an afterthought.

This is why I don't believe that geometry should be taught on the basis of Hilbert's axioms or some other improvement of Euclid's approach. But this point of view in no way denigrates the historical importance of Hilbert's work on the foundations of geometry. Not only did Hilbert show that the axiomatic approach due to Euclid and his contemporaries could be improved to suit the criteria of rigor of the 20th century mathematics, but he was the first to put in practice the fundamental idea of axiomatic mathematics as the science that studies... undefined objects!

Answers & Hints

Here the reader should **not** expect to find solutions to the exercises, only answers (without any comments) to those where it is required to compute or find something, and hints for those (the majority) that begin with the words "Prove that...". The hints are never detailed and are written in a very informal style (not to be imitated by the student in his/her written homework assignments, if such are required by the instructor!); they usually only indicate the strategy of the proof, or some of its ingredients, or vaguely indicate what sort of proof can be obtained. In the latter case we write "*Not very helpful Hint*". Note also that practically all the exercises in the first nine chapters are supplied with answers or hints, whereas starting from Chapter 10, most of the exercises are not. This is because the author feels that by then, when the student has worked through half the book, the time has come to throw him/her into the pool to see if he/she is able to swim unassisted.

Chapter 1

1.1. There are three rotations (of orders 1, 3, 3) and three reflections in the altitudes of the triangle (all of order 2), and four nontrivial subgroups (three groups of order 2 and one of order 3); the group of motions has 3 elements.

1.2. (a) The symmetry group of such a pyramid is isomorphic to that of the square. It has 8 elements (five of order 2, two of order 4, and one of order 1), 6 nontrivial subgroups (five of order 2, one of order 4); there are 4 elements in the group of motions.

(b) The group is isomorphic to the permutation group S_4 (see Chapter 2). There are 24 elements of orders 1, 2, 3, of which 12 are rotations, 12 are reflections w.r.t. planes (of two different types), there are 19 nontrivial subgroups (fifteen of order 2 and four of order 3), and the motion group has 12 elements.

(c) There are 48 elements of orders 1, 2, 3, 4, of which 24 are rotations (see 1.2.3) and 24 are orientation-reversing transformations; the motion group has 24 elements.

(d)* There are 120 elements of orders 1, 2, 3, 4, 5, of which 60 are rotations and 60 are orientation-reversing transformations; the motion group has 60 elements.

(e)* The symmetry group of the icosahedron is isomorphic to that of the dodecahedron, and so the answers are the same as for (d).

(f) There are n rotations, n reflections (of two different types if n is even); the nontrivial subgroups are the rotation subgroup of order n (which has nontrivial subgroups whose orders divide n), the subgroups isomorphic to the dihedral group D_m (see Chapter 3) for any m that divides n, and n reflection subgroups (of order 2); the group of motions has n elements.

1.3. It suffices to place the square onto one of the faces of the cube (respectively, the circle on the equator of the sphere) and check that the different motions of that face (resp., of the equator) can be extended to different motions of the cube (resp., of the sphere).

1.4. Whenever n divides m.

1.5. There are 7 such subgroups, of which 4 are rotations.

1.6. The four main diagonals.

1.7. (a) One reflection in the plane passing through an edge and the midpoint of the opposite edge and one rotation about the line joining the midpoints of two opposite edges.

(b) A reflection in a plane passing through the parallel diagonals of opposite faces and the composition of another such reflection with the central symmetry w.r.t. the center of the cube.

1.8. (a, b, c) For example, any tetrahedron whose vertices are: a vertex A of the polyhedron, the midpoint of an edge AB, the center of a face containing AB, and the center of the polyhedron itself.

1.9. From the unit sphere in $\mathbb{R}^3$, cut off two small symmetric caps by planes parallel to the coordinate plane Oxy; then the set of lines passing through the origin and through what is left of the sphere is the Möbius strip.

1.10. The axis of rotation is the intersection of the planes, while the angle of rotation is twice the angle between the planes.

1.11. It is the rotation whose axis of rotation is the intersection of the two planes passing through each of the two given axes of rotation and making angles equal to half the angle of the corresponding rotation with the plane containing both axes of rotation.

Chapter 2

2.1. $\mathbb{Z}_2$ (symmetries of the unit interval), $\mathbb{Z}_3$ (rotations of the equilateral triangle), $\mathbb{Z}_4$ (rotations of the square), the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (symmetries of the rectangle), $\mathbb{Z}_5$ (rotations of the regular pentagon), S_3 (symmetries of the equilateral triangle), $\mathbb{Z}_6$ (rotations of the regular hexagon).

2.2. (a) The only normal subgroup of $\mathrm{Sym}(\triangle)$ is its rotation subgroup, and the corresponding quotient group is isomorphic to $\mathbb{Z}_2$.

(b) The subgroup of all the motions and the 4-element subgroup (isomorphic to the Klein group) consisting of the identity and the three rotations by π about the three lines joining the midpoints of two opposite edges.

2.3. *Not very helpful Hint.* The proof is a straightforward verification of definitions.

2.4. *Hint.* If H is a subgroup of G, we must prove that for all $g \in G$ we have $g^{-1}Hg \subset H$; take any $h \in H$, consider two cases ($hg \in H$ and $hg \notin H$), and prove the required inclusion – it is obvious in the first case and follows in the second case because there are only two cosets mod H.

2.5. The orbits are the cycles

$$(1 \to 5 \to 4 \to 9 \to 1),\ (2 \to 8 \to 2),\ (3),\ (6 \to 10 \to 7 \to 6),$$

while the stabilizers of elements of the first orbit are $\{1, a^4, a^8\}$, of the second one, $\{1, a^2, \dots, \}$, of the third, the whole group, of the fourth, $\{1, a^3, a^6, a^9\}$, where a is the given permutation.

2.6. (a) 6; (b) 60.

2.7. 16.

2.8. *Hint.* First prove that S_n is generated by permutations σ_i of successive elements (i and $i+1$) and then that any σ_i can be obtained by conjugating $(1\,2)$ by an appropriate power of $(1, 2, \dots, n)$.

2.9. $\langle s_1, s_2 : s_1 s_2 s_1 = s_2 s_1 s_2, s_1^2 = 1, s_2^2 = 1\rangle$; another possibility is $\langle p, q : p^2 = 1, q^3 = 1, (pq)^2 = 1\rangle$; there are several more.

2.10. 36, including 18 epimorphisms.

2.11. An isomorphism between the given group and D_n may be constructed by assigning a to one of the rotations by π and assigning b to the rotation by $2\pi/n$.

2.12. *Not very useful Hint.* There are many computations showing that $a = 1$: one should start with the given formula $b^{-1}ab = a^2$ and appropriately use the trivial relations and the relations $a^5 = b^3 = 1$.

Chapter 3

3.1. The symmetry group is the dihedral group $\mathbb{D}_6$, the motion group is $\mathbb{Z}_6$, both are finite subgroups of O(3) in accordance with Corollary 3.2.8.

3.2. (a) $\widetilde{\mathbb{D}_6}$ and $\mathbb{Z}_6$;
(b) $\mathbb{D}_5$ and $\mathbb{Z}_5$;
(c) $\widetilde{\mathbb{D}_6}$ and $\mathbb{Z}_6$.

3.3. *Hint.* Try to understand the meaning of the two terms on the right-hand side in terms of the number of points in the orbits and the number of elements in the stabilizers in the case of the motion group of the cube. The meaning is the same in the general case and you will see that the equation is practically a tautology – it expresses two ways to compute the number of elements in F.

3.4. Yes, any group of isometries leaving in place an inscribed regular tetrahedron.

3.5. No, because 60 is not divisible by 24.

3.6. There are 6 subgroups isomorphic to $\mathbb{Z}_2$, 4 to $\mathbb{Z}_3$, and 3 to $\mathbb{D}_4$.

3.7. Let M_1N_1 and V_2N_2 be the sides of the pentagons placed on top of the cube (see Figure 3.6) and let Π be the vertical plane parallel to M_1N_1 and V_2N_2 that cuts the cube into two congruent halves. Now rotate the two pentagons around the sides AD and BC until M_1N_1 and M_2N_2 meet Π. From considerations of symmetry it is easy to see that the sides M_1N_1 and V_2N_2 will merge: $M_1 = M_2 =: M$ and $N_1 = N_2 =: N$. Denote by P the midpoint of MN, by S the midpoint of BC, by R the midpoint of CD. Extend the line MR by the segment RQ of length equal to $|PS|$ and let K be the projection of Q on the front face of the cube. Let H be the projection of P on the top face of the cube. It now suffices to prove that the triangles RKQ and SHP are congruent, but this is easy.

3.8. A straightforward verification shows that G^+ does act on F. There are three orbits (one for each of the three different types of points of F, which correspond to faces, edges, and vertices), the stabilizers of the points corresponding to the faces, edges, and vertices consist of 3, 1, and 2 elements, respectively; these elements are rotations by angles that are integer multiples of $\pi/2$, π, and $2\pi/3$,

respectively. The vertices of the cube are those elements of F whose stabilizers have exactly two elements.

3.9. This exercise is similar to Problem 3.8, but somewhat simpler.

3.10. This exercise is similar to Problem 3.8, but a bit more complicated.

3.11. *Hint.* The proof is similar to that sketched in Problem 3.8, except that for the vertices of the octahedron one chooses those elements of F whose stabilizers have exactly three elements.

Chapter 4

4.1. *Hint.* Let $A'B'$ be the image of the segment AB under the given motion. If the lines AB and $A'B'$ are parallel, then it is easy to show that the motion is a translation by the vector $\overrightarrow{AA'}$. Otherwise it is a rotation whose center can be easily constructed.

4.2. *Hint.* Let $A'B'$ be the image of AB under the given isometry. By parallel translation, move $A'B'$ to $A''B''$, where $A'' \equiv A$. The glide symmetry line will then be a line that passes through the midpoint of AA' and is parallel to the bisector of angle BAB''.

4.3. *Not very helpful Hint.* The validity of the construction basically follows from the fact that the sum of the angles of a Euclidean triangle is equal to π. When $\varphi = \psi$, one obviously obtains the parallel translation by the vector joining the the first center of rotation to the second.

4.4. *Hint.* The center is obtained from the center of the given rotation by shifting it by minus the given translation vector.

4.5. *Not very helpful Hint.* The composition is obviously the rotation indicated in the statement of the exercise.

4.6. (a) Two parallel translations.

(b) Two parallel translations and one rotation by the angle π.

(c) Two perpendicular parallel translations by two squares and one rotation by π.

(d) Two rotations by $2\pi/3$ and two parallel translations.

(e) Two rotations by $2\pi/3$ and two parallel translations.

(f) Two reflections and two translations (by the doubled sides of the rectangles).

4.7. No.

4.8. There are two types: first, the vertices of the squares near which two dots of the question marks are located (the corresponding subgroups are rotations by π), and, second, those in the vicinity of which there are no dots (the corresponding subgroups are rotations by $\pi/2$).

4.9. For example, the group preserving the square lattice can be presented as follows: $\langle h, v, r : hvh^{-1}v^{-1} = 1, rvr^{-1}v^{-1=1}, r^4 = 1\rangle$.

4.10. Cubes only.

4.11. The picture on the left corresponds to the discrete geometry shown in Figure 4.5(a), the one on the right, to the middle picture in the second row of Figure 4.6.

4.12. The first from the left in the second row of Figure 4.6.

4.13. The one pictured in Figure 4.5(c) and the middle one in the last row of Figure 4.6.

4.14. The one pictured in Figure 4.5(c) and the middle ones in the first and third rows of Figure 4.6.

4.15. The ones pictured in Figure 4.5(b), (f), the ones to the right in the first and second rows of Figure 4.6, and the one to the left in the third row.

4.16. Rotate all the question marks in (c) so that they lie horizontally with the dot to the right.

Chapter 5

5.1. When both angles are rational multiples of π. In that case, let $\alpha = k\pi/p$ and $\beta = l\pi/q$, where k and p, as well as l and q, are coprime. Then for the fundamental domain one may take any dihedral angle containing the z-axis whose measure equals $(\gcd(k,l)/\mathrm{lcm}(p,q))\cdot\pi$.

5.2. (a) If and only if the triangle is one of the three Coxeter triangles.

(b) The triangle itself will be a fundamental domain.

5.3. Yes, it does determine a Coxeter geometry, namely the one with fundamental domain, the equilateral triangle.

5.4. For example, in the case of the equilateral triangle and of the vertex $s_1 \cap s_2$, they are the two reflections s_1 and s_2 and the two rotations by $2\pi/3$ and $4\pi/3$, i.e., $s_1 \circ s_2$ and $(s_1 \circ s_2)^2$.

5.6. *Hint.* This fact can be proved by inspection of the seven Coxeter polyhedra or directly, by using the definition of the Coxeter dihedral angles.

5.7. *Not very helpful Hint.* (a) Obvious. (b) Follows from (a).

5.8. For example, $\widetilde{B_5}$ is an irregular polytope with 5 faces; one face forms dihedral angles of $\pi/4$ with two others, one of these two forms dihedral angles of $\pi/3$ with the remaining two faces, which are orthogonal to each other.

5.9. (a) No. (b) Yes. (c) No. (d) No.

Chapter 6

6.1. *Not very helpful Hint.* The proof is an exercise in space geometry of the same type as the proof of the spherical sine theorem.

6.2. *Not very helpful Hint.* The proof is an exercise in space geometry of the same type as the proof of the spherical sine theorem.

6.3. *Hint.* If ABC is the given triangle, consider the tetrahedron $OABC$, where O is the center of the sphere and make use of the fact that the three points A, B, C all lie in one half-sphere.

6.4. It doesn't. The analog immediately follows from the corresponding cosine theorems.

6.5. The geodesic between Moscow and New York intersects Greenland but not Spain.

6.6. π and 3π.

6.7. *Hint.* Use the spherical analog of the Pythagorean theorem.

6.8. *Hint.* Use r to compute the Euclidean radius of the circle and then plug it into the classical formula for the area of a spherical cap.

6.9. *Hint.* For example, for the cube, a fundamental domain is given by the pyramid $OIAH$, where O is the center of symmetry of the cube, I is the center of the bottom square $ABCD$, and H is the foot of the perpendicular drawn from I to AB. A total of 48 copies of the fundamental domain fill the cube – this can be established without counting the number of pyramids that actually yield such a filling.

6.10. *Hint.* Construct the inscribed and circumscribed circles to triangle ABC regarded as a Euclidean triangle in the plane ABC and then project them on the sphere from its center O.

6.11. *Hint.* Read the hint to the previous exercise.

Chapter 7

7.1. *Hint.* One can compute the equation of the image of the given circle regarded as lying in $\overline{\mathbb{C}}$ by plugging in $z = 1/\overline{z}$ into its equation. There are also purely geometric solutions, mostly using the congruence of right triangles.

7.2. *Hint.* Let M be an arbitrary point of the given circle $\mathcal{C}$ orthogonal to the circle of inversion $\mathcal{C}_O$ of radius 1 centered at O. Let S be the intersection point of the perpendicular to OM with $\mathcal{C}_O$ and N the intersection of the line OM with $\mathcal{C}$. Then the three right triangles OSM, SNM, and ONS are similar, and one readily shows that $|OM| \cdot |ON| = 1$.

7.3. *Hint.* The proof uses the similitude of right triangles in the spirit of the previous exercise.

7.4. *Hint.* The proof uses the similitude of right triangles in the spirit of the two previous exercises.

7.5. *Hint.* The proof uses the similitude of right triangles in the spirit of the three previous exercises.

7.6. *Hint.* Let P be a point inside the given circle $\mathcal{C}$, and M, N the intersection points of line OP with $\mathcal{C}$. Then $|OM| \cdot |ON| = 1$ and this fact easily yields the required bijectivity.

7.7. *Hint.* Consider any hyperbolic isometry α that takes the (Euclidean!) center of the given circle $\mathcal{C}$ to the center O of the disk model. Then $\alpha(\mathcal{C})$ is both a hyperbolic and a Euclidean circle and therefore so is $\alpha^{-1}(\alpha(\mathcal{C}))$. No, in general the centers of the two types do not coincide (they do only if the circle is centered at O).

7.8. *Hint.* Here $\mathbb{H}^2$ is tiled both by regular heptagons and equilateral triangles, each heptagon containing 7 triangles. Those triangles determine a Coxeter geometry. The triangles have angles of $2\pi/7$, the angles at the vertices of the heptagons are of $4\pi/7$.

7.9. *Hint.* This can be checked by a direct computation for the inversion $z \mapsto p/\overline{z}$, $p > 0$, and then for an arbitrary inversion by conjugating it with the parallel shift taking the center of inversion to the point $(0, 0) \in \mathbb{C}$.

7.10. *Hint.* This is related to the cross ratio of four points and is not easy. For the answer, see the next chapter.

7.11. *Hint.* One of the numerous ways of taking the flag (A, l, Π) to the flag (A', l', Π') is, first, to use a (hyperbolic) isometry α that

takes A to A', then an isometry β that takes A' to the center O of the disk; then $\beta(\alpha(l))$ and $\beta(l')$ will be diameters of the disk and a symmetry γ w.r.t. a bisector of the angle formed by these two diameters will interchange them: now the map $\phi := \beta^{-1} \circ \gamma \circ \beta \circ \alpha$ takes A to A' and l to l'; if $\phi(\Pi)$ coincides with the "half-plane" Π', we are done, otherwise the composition of ϕ followed by the symmetry w.r.t. l' is the required isometry.

7.12. *Hint.* Consider a small regular pentagon with center of symmetry at the center of the disk $\mathbb{H}^2$. As the size of the pentagon increases, its angles decrease and, by continuity, at some moment will equal $2\pi/5$. That pentagon will be the fundamental tile. The other tiles are successively added so as to obtain 5 tiles at each vertex.

7.13. *Not very helpful Hint.* This theory is similar to, and not much more complicated than, the two-dimensional theory of inversion described at the beginning of Chapter 7.

7.14. *Hint.* Use the fact that any circle is the intersection of two spheres and any straight line is the intersection of two planes.

7.15. *Hint.* Use the appropriate facts from the three previous exercises and a construction similar to the one in Problem 7.3.

7.16. *Hint.* In this model, planes are the intersections of spheres orthogonal to the *absolute* (i.e., the boundary sphere of the ball) with the ball, the lines are circles orthogonal to the absolute, the isometries are compositions of reflections with respect to the hyperbolic planes.

7.17. *Not very helpful Hint.* The problem easily reduces to a mildly difficult problem in elementary Euclidean plane geometry about orthogonal circles.

7.18. *Hint.* This is also a problem in the geometry of circles with many possible solutions. One of them consists of constructing the common perpendicular to the given two lines, taking its midpoint N and drawing a third parallel n through N, then transforming n by an isometry α to a line n_0 passing through A_∞ and M, and finally using the symmetry of the whole picture w.r.t. n_0 and using α^{-1} to complete the construction.

7.19. *Hint.* Your first guess about the answer will probably be correct!

Chapter 8

8.1. *Not very helpful Hint.* Both (a) and (b) can be proved by direct calculations.

8.2. *Hint.* Let K be the intersection of the perpendicular to AP through its midpoint M with the line l. One of the required circles (namely, the one passing through A) is centered at K and $|KP|$ is its radius.

8.3. *Hint.* By using the transformation Ω^{-1} (see 8.1.6) one can pass from the given situation to the Poincaré disk model, where this problem reduces to constructing the two parallels through a given point to a given line, and then return our situation via Ω. Of course there is also a direct proof in the spirit of the previous exercise.

8.4. *Hint.* First check that the given transformation takes the disk $\mathbb{H}^2$ to itself, then verify that the given formula holds whenever it corresponds to an even number of reflections w.r.t. circles orthogonal to the absolute.

8.5. *Hint.* The hint to the solution of 7.11 applies here as well.

8.6. $\pi - \alpha - \beta - \gamma$, where α, β, γ are the angles of the triangle.

Chapter 9

9.1. *Not very helpful Hint.* Use the formula for the distance function.

9.2. *Not very helpful Hint.* Argue by contradiction.

9.3. *Hint.* This follows immediately from the uniqueness of the perpendicular from a given point to a given line and the definition of reflections.

9.4. *Hint.* Consider the composition of the reflections in the two lines and use Problem 9.3.

9.5. *Hint.* Consider the reflection in one of the perpendiculars and then in the other.

9.6. *Hint.* Argue by contradiction and, using the properties of the model that are also true in Euclidean geometry (such as the properties of the distance listed in 9.1.2 and the existence and uniqueness of perpendiculars) show that Euclid's Fifth Postulate would then hold.

9.7. *Hint.* This is impossible: the existence of a linear space structure in the model means that similar but noncongruent triangles exist, but this, as is not hard to see, is equivalent to Euclid's Fifth Postulate.

9.8. *Hint.* Take an equilateral triangle (in the Euclidean sense) with vertices on the absolute. Then another equilateral triangle in its interior with vertices sufficiently close to the absolute will do.

Chapter 10

10.8. *Hint.* Take a very small equilateral triangle ABC (whose angles are close to $\pi/3$) with center of symmetry at the center O of the Poincaré disk model, consider the homothety with center O and a huge coefficient, and look at the angles of the image of ABC.

10.9. *Hint.* (a) The proof is a nontrivial calculation; see [16], pp. 149–151. (b) Use the previous formula and the formula

$$e^d = \sinh(d) + \cosh(d),$$

similar to Euler's famous formula for e^{-d} and just as easy to prove.

Chapter 12

12.9. Brianchon's theorem may be stateda as follows: Let a, b, c, d, e, f be tangents to a conic. Let l, m, n be the lines passing through

the points $L_1 = a \cap b$ and $L_2 = e \cap d$, $M_1 = a \cap f$ and $M_2 = c \cap d$, $N_1 = c \cap b$ and $N_2 = e \cap f$, respectively. Then the three lines l, m, n intersect at one point. The proof may be obtained by dualizing the proof of Pascal's Theorem.

Chapter 13

13.1. *Not very helpful Hint.* A proof can be obtained by a straightforward calculation in coordinates.

13.5. *Hint.* By choosing a triangle of area less than ε in the southern hemisphere close enough to the equator, we can ensure that the area of its image under the central projection is as large as we wish.

Chapter 14

14.1. The linear spaces $A\mathbb{F}(2)$ and $A\mathbb{F}(3)$ over the fields of 2 and 3 elements may be regarded as affine geometries with 4 and 9 elements, respectively; but they can be constructed without appealing to finite fields. Thus the one with four points a, b, c, d consists of six lines ab, bc, cd, da, ac, bd, three pairs of which (ab and cd, dc and da, ac and bd) are parallel. It can also be described as the Fano projective plane with the "points at infinity" removed. The one with 9 points can also be constructed directly; the reader will profit by drawing a picture of $A\mathbb{F}(3)$ in the style of Figure 14.1.

Chapter 15

15.7. If it was, spherical lines would be subsets of projective lines, and any pair of spherical lines should have one common point or less.

Chapter 16

16.1. $w_{\mathrm{lcm}(k,l)}$.

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Index

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9 780821 875711

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