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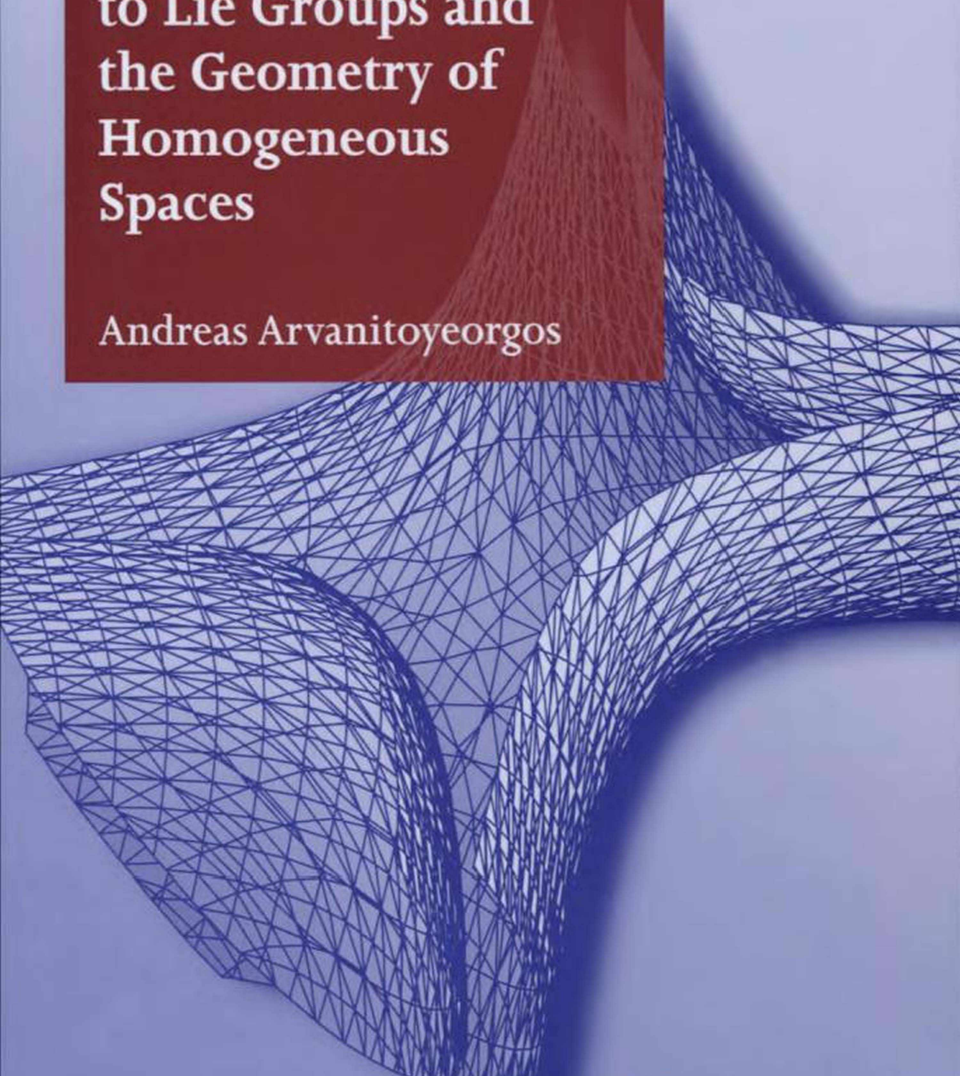
An Introduction to Lie Groups and the Geometry of Homogeneous Spaces

Andreas Arvanitoyeorgos



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Preface

The roots of this book lie in a series of lectures that I presented at the University of Ioannina, in the summer of 1997. The central theme is the geometry of Lie groups and homogeneous spaces. These are notions which are widely used in differential geometry, algebraic topology, harmonic analysis and mathematical physics. There is no doubt that there are several books on Lie groups and Lie algebras, which exhaust these topics thoroughly. Also, homogeneous spaces are occasionally tackled in more advanced textbooks of differential geometry.

The present book is designed to provide an introduction to several aspects of the geometry of Lie groups and homogeneous spaces, without becoming too detailed. The aim was to deliver an exposition at a relatively quick pace, where the fundamental ideas are emphasized. Several proofs are provided, when it is necessary to shed light on the various techniques involved. However, I did not hesitate to mention more difficult but relevant theorems without proof, in appropriate places. There are several references cited, that the reader can consult for more details.

The audience I have in mind is advanced undergraduate or graduate students. A first course in differential geometry would be desirable, but is not essential since several concepts are presented. Also, researchers from neighboring fields will have the chance to discover a pleasant introduction on a variety of topics about Lie groups, homogeneous spaces and related applications.

I would like to express my sincere thanks to the editors for their thorough suggestions on the manuscript, as well as my gratitude to Professors Jurgen Berndt, Martin A. Guest, Lieven Vanhecke, and McKenzie Y. Wang for their kindness in making comments on it.

Andreas Arvanitoyeorgos

Athens, August 2003

Introduction

There are several terms which are included in the title of this book, such as "Lie groups", "geometry", and "homogeneous spaces", so it maybe worthwhile to provide an explanation about their relationships. We will start with the term "geometry", which most readers are familiar with.

Geometry comes from the Greek word "$\gamma\epsilon\omega\mu\epsilon\tau\rho\epsilon\acute{\iota}\nu$", which means to measure land. Various techniques for this purpose, including other practical calculations, were developed by the Babylonians, Egyptians, and Indians. Beginning around 500 BC, an amazing development was accomplished, whereby Greek thinkers abstracted a set of definitions, postulates, and axioms from the existing geometric knowledge, and showed that the rest of the entire body of geometry could be deduced from these. This process led to the creation of the book by Euclid entitled *The Elements.* This is what we refer to as *Euclidean geometry.*

However, the fifth postulate of Euclid (the parallel postulate) attracted the attention of several mathematicians, basically because there was a feeling that it would be possible to prove it by using the first four postulates. As a result of this, new geometries appeared (elliptic, hyperbolic), in the sense that they are consistent without using Euclid's fifth postulate. These geometries are known as *Non-Euclidean Geometries*, and some of the mathematicians that contributed to their development were N. I. Lobachevsky, J. Bolyai, C. F. Gauss, and E. Beltrami.

A detailed theory of surfaces in three-dimensional space was developed by C. F. Gauss. His main result was the *Theorema Egregium*, which states that the curvature of a surface is an "intrinsic" property of the surface. This means it can be measured and "felt" by someone who is on the surface, rather than only by observing the surface from outside.

However, the fundamental question "What is geometry?" still remained. There are two directions of development after Gauss. The first, is related to the work of B. Riemann, who conceived a framework of generalizing the theory of surfaces of Gauss, from two to several dimensions. The new objects are called *Riemannian manifolds*, where a notion of curvature is defined, and is allowed to vary from point to point, as in the case of a surface. Riemann brought the power of calculus into geometry in an emphatic way as he introduced metrics on the spaces of tangent vectors. The result is today called *differential geometry.*

The other direction is the one developed by F. Klein, who used the notion of a transformation group to define geometry. According to Klein, the objects of study in geometry are the invariant properties of geometrical figures under the actions of specific transformation groups. Hence, the consideration of different transformation groups leads to different kinds of geometry, such as Euclidean geometry, affine geometry, or projective geometry. For example, Euclidean geometry is the study of those properties of the plane that remain invariant under the group of rigid motions of the plane (the Euclidean group). The groups that were available at that time, and which Klein used to determine various geometries, were developed by the Norwegian mathematician Sophus Lie, and are now called *Lie groups.*

This brings us to the other terms of the title of this book, namely "Lie groups" and "homogeneous spaces" The theory of Lie has its roots in the study of symmetries of systems of differential equations, and the integration techniques for them. At that time, Lie had called these symmetries "continuous groups" In fact, his main goal was to develop an analogue of Galois theory for differential equations.

The equations that Lie studied are now known as equations of Lie type, and an example of these is the well-known Riccati equation. Lie developed a method of solving these equations that is related to the process of "solution by quadrature" (cf. [**Fr-Uh, pp. 14, 55**], [**Ku**]). In Galois' terms, for a solution of a polynomial equation with radicals, there is a corresponding finite group. Correspondingly, to a solution of a differential equation of Lie type by quadrature, there is a corresponding continuous group.

The term "Lie group" is generally attributed to E. Cartan (1930). It is defined as a manifold G endowed with a group structure, such that the maps $G \times G \to G$ $(x, y) \mapsto xy$ and $G \to G$ $x \mapsto x^{-1}$ are smooth (i.e. differentiable). The simplest examples of Lie groups are the groups of isometries of $\mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{H}^n$ ($\mathbb{H}$ is the set of quaternions). Hence, we obtain the orthogonal group $O(n)$, the unitary group $U(n)$, and the symplectic group $Sp(n)$.

An algebra $\mathfrak{g}$ can be associated with each Lie group G in a natural way; this is called the *Lie algebra* of G. In the early development of the theory, $\mathfrak{g}$ was referred to as an "infinitesimal group" The modern term is attributed by most people to H. Weyl (1934). A fundamental theorem of Lie states that every Lie group G (in general, a complicated non-linear object) is "almost" determined by its Lie algbera $\mathfrak{g}$ (a simpler, linear object). Thus, various calculations concering G are reduced to algebraic (but often non-trivial) computations on $\mathfrak{g}$.

A *homogeneous space* is a manifold M on which a Lie group acts transitively. As a consequense of this, M is diffeomorphic to the coset space G/K, where K is a (closed Lie) subgroup of G. In fact, if we fix a base point $m \in M$, then K is the subgroup of G that consists of the points in G that fix m (it is called the *isotropy subgroup of* m). As mentioned above, these are the geometries according to Klein, in the sense that they are obtained from a manifold M and a transitive action of a Lie group G on M. The advantage is that instead of studying a geometry with base point m as the pair (M, m) with the group G acting on M, we could equally study the pair (G, K).

One of the fundamental properties of a homogeneous space is that, if we know the value of a geometrical quantity (e.g. curvature) at a given point, then we can calculate the value of this quantity at any other point of G/K by using certain maps (translations). Hence, all calculations reduce to a single point which, for simplicity, can be chosen to be the identity coset $o = eK \in G/K$. Furthermore, in an important special case where the homogeneous space is *reductive*, then the tangent space of G/K at o can be identified in a natural way with a subspace of $\mathfrak{g}$.

As a consequence of this, many hard problems in homogeneous geometry can be formulated in terms of the group G and the subgroup K, and then in terms of their corresponding infinitesimal objects $\mathfrak{g}$ and $\mathfrak{k}$. Such an infinitesimal approach enables us to use linear algebra to tackle non-linear problems (from geometry, analysis, or theory of differential equations). For example, the equations satisfied by an Einstein metric (these, according to general relativity, describe the evolution of the universe) are a complicated non-linear system of partial differential equations. However, for G-invariant metrics on a homogeneous space, this system reduces to a system of algebraic equations, which can be solved in many cases.

There is a large variety of applications of Lie groups in mathematics. They appear in various ways beyond differential geometry, such as algebraic topology, harmonic analysis, and differential equations, to name a few. They also possess important applications in physics, since they become involved in field theories in many ways. In fact, certain classical Lie groups appear as the building blocks in various physical theories of matter. Homogeneous spaces, in turn, have been employed in the physics of elementary particles as models called *supersymmetric sigma models*. Also, what physicists call *coherent states*, are in one-to-one correspondence with elements in a homogeneous space.

Before we proceed to the description of the chapters of this book, we would like to mention that the two generalizations of Euclidean geometry that we mentioned, namely that of Riemann and that of Klein, were unified by E. Cartan in his theory of *espaces généralizés*. In Cartan's geometry, at each point m of M, there is a Klein-style geometry in the tangent space. That is to say, Cartan took Klein's geometry and made it local to each tangent space.

Chapter 1 starts with a simple example of a Lie group that exhibits the manifold and group structure. Then we give a brief review of manifolds, and then we proceed with the definition of a Lie group. We define the Lie algebra of a Lie group as the tangent space at the identity element of the group, and alternatively as the set of its one-parameter subgroups. We also list a simplified version of Lie's theorems.

In Chapter 2, after discussing a few elementary concepts about representations, we develop the appropriate tools that are needed for the classification of the compact and connected Lie groups. These are the adjoint representation, and the maximal torus of a Lie group. We also introduce a very useful tool, the Killing form, and we provide a brief insight through the complex semisimple Lie algebras.

Chapter 3 starts with a brief review of Riemannian manifolds, and then discusses a way to make a Lie group into a Riemannian manifold. The metrics which are important here are the bi-invariant metrics, and with respect to such metrics we give formulas for the connection and the various types of curvatures.

In Chapter 4 we define the notion of a homogeneous space and provide several examples. We discuss the reductive homogeneous spaces, and the isotropy representation of such a space.

The geometry of a homogeneous space is discussed in Chapter 5, where we show how a homogeneous space G/K can become a Riemannian manifold (so we obtain a *Riemannian homogeneous space*). The important metrics here are the G-invariant metrics. Formulas are presented for the connection and the various types of curvatures.

In Chapters 6 and 7 we discuss two important, and generally non-overlapping, classes of homogeneous spaces, which are the symmetric spaces and the generalized flag manifolds. One of the most significant advances of the twentieth century mathematics is Cartan's classification of semisimple Lie groups. This leads to the classification of these two classes of homogeneous spaces. These spaces have many applications in real and complex analysis, topology, geometry, dynamical systems, and physics.

In Chapter 8 we give three applications of homogeneous spaces. The first is about homogeneous Einstein metrics. These are Riemannian metrics whose Ricci tensor is proportional to the metric. The second refers to symplectic geometry, which is rooted in Hamilton's laws of optics. Here we present a Hamiltonian system on generalized flag manifolds. A Hamiltonian system is a special case of an integrable system, which is a subject that has attracted much attention recently. The third application deals with homogeneous geodesics in homogeneous spaces. Geodesics are important not only in geometry, being length minimizing curves, but also have important applications in mechanics since, for example, the equation of motion of many systems reduces to the geodesic equation in an appropriate Riemannian manifold. Here, we present some results about homogeneous spaces, all of whose geodesics are homogeneous, that is, they are orbits of one-parameter subgroups. These are usually known in the literature as g.o. spaces.

Chapter 1

Lie Groups

1. An example of a Lie group

A Lie group is a set that has both a manifold and a group structure, which are compatible. So, we will begin this discussion with an example that exhibits these two properties.

Let $\mathrm{M}_n\mathbb{R}$ be the set of all $n \times n$ real matrices. We associate to the matrix $A = (a_{ij})$ the point in the Euclidean space $\mathbb{R}^{n^2}$ whose coordinates are $a_{11}, a_{12}, \ldots, a_{nn}$. Hence, topologically $\mathrm{M}_n\mathbb{R}$ is simply the Euclidean n^2 space. Next we define the *general linear group* $\mathrm{GL}_n\mathbb{R}$ to be the group (under usual matrix multiplication) of all $n \times n$ real matrices $A = (a_{ij})$ with determinant $\det A \neq 0$. Since $\det A$ is a polynomial of degree n in the coordinates, it is a smooth function on $\mathrm{M}_n\mathbb{R}$. Furthermore, since the set $\mathbb{R} \setminus \{0\}$ forms an open set in $\mathbb{R}$, and since the inverse image of an open set under a continuous map is open, the set $\mathrm{GL}_n\mathbb{R}$ is an open subset of $\mathrm{M}_n\mathbb{R}$. Hence, topologically $\mathrm{GL}_n\mathbb{R}$ is an open subset of a Euclidean space, and as such is an n^2-dimensional *manifold*, as will be seen later on. This takes care of the manifold and the group structure structure of $\mathrm{GL}_n\mathbb{R}$. Let us now see how they interact.

Since $(ab)_{ij} = \sum a_{ik}b_{kj}$, the product matrix AB has coordinates that are smooth functions of the coordinates of A and B. Also, from the formula for the inverse

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}A$$

(where $\operatorname{adj}A$ is the matrix whose entries are the signed cofactors of each of the entries a_{ij}), we see that the coordinates of A^{-1} are also smooth functions of those of A. This concludes the description of the general linear group $\mathrm{GL}_n\mathbb{R}$ as a manifold and as a group, with the group operations of multiplication and inverse being smooth functions. It is an important example of a *Lie group*. We will see more examples of Lie groups later on, after we make a brief review of various definitions, notations and results about manifolds which will be used later on.

2. Smooth manifolds: A review

Generally speaking, a smooth manifold is a topological space M that locally resembles the Euclidean space $\mathbb{R}^n$, with a notion of differentiation that can be established in M. The formal definition is as follows:

Definition. A smooth (or differentiable) *manifold* of dimension n is a Hausdorff topological space M with a collection of pairs (U_α, ϕ_α) where U_α (*chart*) is an open subset of M and $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n$ so that:

(a) Each ϕ_a is a homeomorphism of U_α onto an open subset V_α of $\mathbb{R}^n$

(b) $\cup_\alpha U_\alpha = M$.

(c) For every α, β the transition functions $\phi_{\alpha\beta} = \phi_\beta \circ \psi_a^{-1} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth, in the sense of smooth functions between subsets of $\mathbb{R}^n$. In this case the charts (U_α, ϕ_α) and (U_β, ϕ_β) are called compatible.

(d) The family $\{(U_\alpha, \phi_a)\}$ is maximal relative to the conditions (b) and (c).

Such a family of sets and maps satisfying (b), (c), and (d) constitutes a *smooth structure* on M.

Remark. Condition (d) is a purely technical one. Given a family of charts satisfying (a)-(c) it can be completed to a maximal one, by taking the union of all charts that, together with any of the ones originally chosen, satisfy condition (c). Hence, with a certain abuse of language, we say that a smooth manifold is a set that satisfies conditions (a)-(c), and the extension to the maximal atlas is done without further comment.

Examples.

(1) The Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold, covered by only one chart $U = \mathbb{R}^n$, $\phi\colon U \to \mathbb{R}^n$ the identity map.

(2) The *sphere* $S^n = \{x = (x_1, x_2, \dots x_{n+1}) \in \mathbb{R}^{n+1} \colon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ in $\mathbb{R}^{n+1}$ is a manifold of dimension n. It can be covered by two charts $U_+ = \{x \in S^n \colon x_{n+1} > -1\}$ with $\phi_+\colon U_+ \to R^n$ by $\phi_+(x) = \left(\frac{x_1}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}}\right)$, and $U_- = \{x \in S^n \colon x_{n+1} < 1\}$ with $\phi_-(x) = \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}}\right)$. The maps ϕ_+ and ϕ_- are called *stereographic projections.*

(3) The *projective space* $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ that pass through $0 \in \mathbb{R}^{n+1}$. More precisely, $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation

$$(x_1, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1}), \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

The points of $\mathbb{R}P^n$ will be denoted by $[x_1, \dots, x_{n+1}]$. Define the subsets $U_i = \{[x_1, \dots, x_{n+1}] \colon x_i \neq 0\}$ $(i = 1, \dots, n+1)$ of $\mathbb{R}P^n$. Then the maps $\phi_i \colon U_i \to \mathbb{R}^n$ $(i = 1, \dots, n+1)$ given by

$$\phi([x_1, \dots, x_{n+1}]) = [x_1 x_i^{-1}, \dots, x_{i-1} x_i^{-1}, x_{i+1} x_i^{-1}, \dots, x_{n+1} x_i^{-1}]$$

are 1-1 and onto. The projective space is covered by the charts $(U_1, \phi_1), \dots, (U_{n+1}, \phi_{n+1})$.

(4) Any open subset U of a smooth manifold M is itself a smooth manifold. The charts of U are the intersections of U with the charts of M.

(5) If M and N are smooth manifolds, then the Cartesian product $M \times N$ is also a smooth manifold of dimension equal to the sum of the dimensions of M and N.

By using charts we can define differentiability for functions between smooth manifolds.

Definition. Let M and N be two smooth manifolds and $f\colon M \to N$ a function. Then f is called *smooth* (or differentiable) if for any two charts $\phi\colon U \to V$ and $\widetilde{\phi}\colon \widetilde{U} \to \widetilde{V}$ of M and N respectively, the map

$$\widetilde{\phi} \circ f \circ \phi^{-1}\colon \phi(U \cap f^{-1}(\widetilde{U})) \to \widetilde{V}$$

is a smooth (differentiable) function between Euclidean spaces.

A *diffeomorphism* $f\colon M \to N$ is a smooth function that has an inverse which is also smooth.

Next, we will discuss tangent vectors and vector fields. Let $\mathcal{F}(M)$ be the set of all smooth real-valued functions on a manifold M.

Definition. Let p be a point of a manifold M. A *tangent vector* to M at p is a real-valued function $v\colon \mathcal{F}(M) \to \mathbb{R}$ that satisfies:
(a) $v(af + bg) = av(f) + bv(g)$,
(b) $v(fg) = v(f)g(p) + f(p)v(g)$ (Leibniz rule) $(a, b \in \mathbb{R},\ f, g \in \mathcal{F}(M))$.

At each point $p \in M$ let $T_p(M)$ be the set of all tangent vectors to M at p. Then under the operations

$$(v + w)(f) = v(f) + w(f),$$
$$(av)(f) = av(f),$$

the set $T_p(M)$ is made into a real vector space of dimension equal to that of M. A basis for this vector space is constructed as follows:

Take a local chart (U, ϕ) of p, and let x_i $(i = 1, \ldots, n)$ be the i^{th} component of ϕ (i.e., the result of the composition of $\phi\colon U \to \mathbb{R}^n$ with the i^{th} projection $u_i\colon \mathbb{R}^n \to \mathbb{R}$.) Then the function

$$\left.\frac{\partial}{\partial x_i}\right|_p \quad \mathcal{F}(M) \to \mathbb{R}$$

sending each $f \in \mathcal{F}(M)$ to

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial u_i}(\phi(p))$$

is a tangent vector to M at p, and the set of all such tangent vectors for $i = 1, \ldots, n$ is a basis for $T_p(M)$.

We now set $TM = \cup_p T_p(M)$ the disjoint union over all points of the tangent vectors at each point. Thus a point in this new space consists of a pair (p, v), where p is a point of M and v is a tangent vector to M at the point p. The set TM can be made into a manifold of dimension $2n$, called the *tangent bundle* of M. The map $\pi\colon TM \to M$ given by $\pi(p, v) = p$ $(p \in M,\ v \in T_pM)$ is called the *canonical projection.* The manifold structure on TM is chosen so that π is a smooth map. For each $p \in M$ the pre-image $\pi^{-1}(p)$ is exactly the tangent space T_pM. It is called the *fiber* over p.

A *curve* in a manifold M is a smooth map $\alpha\colon I \to M$, where I is an open interval in $\mathbb{R}$. There are several equivalent ways to define a notion of a *velocity vector* $\alpha'(t)$ of the curve α at t. Here we will adopt the following: The velocity vector of α is the vector $\alpha'(t) \in T_{\alpha(t)}M$ defined by

$$\alpha'(t)f = \frac{d(f \circ \alpha)}{dt}(t)$$

for all $f \in \mathcal{F}(M)$. This definition is motivated from the notion of directional derivative in advanced calculus. Indeed, let $\alpha \quad I \to \mathbb{R}^n$ be a smooth curve in $\mathbb{R}^n$ with $\alpha(0) = p$. Let $\alpha(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$. Then $\alpha'(0) = (x_1'(0), \ldots, x_n'(0)) = v \in \mathbb{R}^n$. Also, let f be a smooth function defined in a neighborhood of p. Then by restricting f to the curve α, the directional derivative with respect to the vector $v \in \mathbb{R}^n$ is

$$vf = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}$$

A curve is a special case of a map between manifolds. The notion of the velocity vector (derivative of the curve) can be extended to smooth functions between manifolds.

Definition. Let $f\colon M \to N$ be a smooth function. Then, for each $p \in M$, the *differential* of f is the function

$$df_p\colon T_pM \to T_{f(p)}N$$

defined by

$$df_p(v)(g) = v(g \circ f)$$

for all $v \in T_pM$ and $g \in \mathcal{F}(N)$.

At each point $p \in M$, the differential df_p is a linear function between the tangent spaces.

The following proposition provides a useful method of computing the differential of a function.

Proposition 1.1. *Let $f\colon M \to N$ be a smooth map between two manifolds, and let $p \in M$ and $v \in T_pM$. Take any smooth curve $\alpha\colon I \to M$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Then the differential of f at p is given by*

$$df_p(v) = \frac{d}{dt}(f \circ \alpha)\bigg|_{t=0}$$

We now come to vector fields. A *vector field* X on a manifold M is a function that assigns to each point $p \in M$ a tangent vector X_p to M at p. Thus $X\colon M \to TM$ with $X_p \in T_pM$. We can think of X as a collection of arrows, one at each point of M. If X is a vector field on M and $f \in \mathcal{F}(M)$, then Xf denotes the real-valued function on M given by

$$Xf(p) = X_p(f) \qquad \text{for all } p \in M.$$

The vector field X is called *smooth* if the function Xf above is smooth for all $f \in \mathcal{F}(M)$. We will denote by $\mathcal{X}(M)$ the set of all smooth vector fields on a manifold M.

Now, the function defined above can be viewed as a map $X\colon \mathcal{F}(M) \to \mathcal{F}(M)$ which sends f to Xf. This map has the properties of a *derivation*, i.e., the following are satisfied:

$$X(af + bg) = aX(f) + bX(g) \qquad a, b \in \mathbb{R},$$
$$X(fg) = X(f)g + fX(g) \qquad \text{(Leibniz rule)}.$$

Conversely, any derivation D on $\mathcal{F}(M)$ comes from a smooth vector field. In fact, for each $p \in M$ define $X_p\colon \mathcal{F}(M) \to \mathbb{R}$ by $X_p(f) = D(f)(p)$. This interpretation of vector fields as derivations leads to an important operation on vector fields.

Let $X, Y \in \mathcal{X}(M)$. Define $[X,Y] = XY - YX$. This is a function from $\mathcal{F}(M)$ to $\mathcal{F}(M)$ sending each f to $X(Yf) - Y(Xf)$. An easy computation shows that $[X,Y]$ is a derivation on $\mathcal{F}(M)$, hence a smooth vector field on M, which is called the *bracket* of X and Y The bracket assigns to each $p \in M$ the tangent vector $[X,Y]_p$ such that

$$[X,Y]_p(f) = X_p(Yf) - Y_p(Xf).$$

Furthermore, the bracket operation has the following properties:

$$\begin{aligned}
&(a)\quad [X,Y] = -[Y,X] \qquad \text{(skew-symmetry)},\\
&(b)\quad [aX+bY,Z] = a[X,Z]+b[Y,Z],\\
&\qquad\ [Z,aX+bY] = a[Z,X]+b[Z,Y] \qquad (\mathbb{R}\text{-bilinearity}),\\
&(c)\quad [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0 \qquad \text{(Jacobi identity)}.
\end{aligned}$$

The above properties say that the set $\mathcal{X}(M)$ with the operation "bracket" of vector fields is a real *Lie algebra*. In general, a real (respectively complex) Lie algebra is a real (respectively complex) vector space V with an operation $[\quad]\colon V \times V \to V$ that satisfies properties (a)-(c) above.

The bracket of vector fields has an interpretation as a derivation of Y along the "flows" of X to be explained now. The following proposition is a manifold version of the existence and uniqueness theorem for ordinary differential equations (see e.g. [**Bo-Di, p. 37**]).

Proposition 1.2. *Let X be a smooth vector field on a smooth manifold M, and let $p \in M$. Then there exists an open neighborhood U of p, an open interval I around 0, and a smooth mapping $\phi : I \times U \to M$ such that the curve $\alpha_q\colon I \to M$ given by $\alpha_q(t) = \phi(t,q)$ $(q \in U)$ is the unique curve that satisfies $\frac{\partial\phi}{\partial t} = X_{\alpha_q(t)}$ and $\alpha_q(0) = q$.*

A curve with the above property is called an *integral curve* of the vector field X. If t is kept constant, the above proposition shows that the assignment $q \mapsto \alpha_q(t)$ defines a function $\phi_t\colon U \to M$ on a neighborhood U of p. This function is called the *local flow* of X. The local flow has the properties:

(*a*) ϕ_0 is the identity map of U,
(*b*) $\phi_s \circ \phi_t = \phi_{s+t}$ for all $s, t \in U$,
(c) each flow is a diffeomorphism with $\phi_t^{-1} = \phi_{-t}$.

The interpretation of the bracket $[X, Y]$ is contained in the following proposition:

Proposition 1.3. *Let X, Y be smooth vector fields on a smooth manifold M, $p \in M$, and ϕ_t the local flow of X in a neighborhood of p. Then*

$$[X, Y]_p = \lim_{t \to 0} \frac{1}{t}[Y_p - d\phi_{-t}(Y_{\phi_t(p)})].$$

The above proposition expresses, in a sense, for each p the rate of change of Y in the direction of X, that is, along the integral curve of the vector field X passing through p.

Finally a note on submanifolds. Roughly speaking, a submanifold of a manifold M is a subset of M that acquires its manifold structure from M. More precisely, we have the following:

Definition. A manifold P is a *submanifold* of the manifold M if
(a) P is a topological subspace of M.
(b) The inclusion map $j \colon P \hookrightarrow M$ is smooth and at each point $p \in P$ its differential dj_p is one-to-one.

In general, a mapping between manifolds that satisfies property (b) is called an *immersion*. In the above definition, if a manifold P is merely a subset of M which satisfies property (b) only, it is called an *immersed submanifold* of M. To conclude the picture, P is called an *imbedding* into M if there exists a one-to-one immersion $\phi \colon P \to M$ such that ϕ is a homeomorphism onto $\phi(P)$.

3. Lie groups

A Lie group is a smooth manifold which is also a group so that the group operations are smooth functions. More specifically we have:

Definition. Let G be a smooth manifold. Then G is called a *Lie group* if:

(a) G is a group.

(b) The group operations $G \times G \to G$, $(x, y) \mapsto xy$ and $G \to G$, $x \mapsto x^{-1}$ are smooth functions.

Examples.

(1) The sets $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ are Lie groups under vector addition. Here $\mathbb{H}$ is the set of *quaternions* that consists of the numbers $q = t + ix + jy + kz$ $(t, x, y, z \in \mathbb{R})$ in $\mathbb{R}^4$ with basis $1, i, j, k$, and commutation relations $i^2 = j^2 = k^2 = -1$, $ij = k, ji = -k, ik = -j, ki = j, kj = -i, jk = i$.

(2) The sets $\mathbb{R}^*, \mathbb{C}^*, \mathbb{H}^*$ are Lie groups under multiplication. (Here $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, etc.).

(3) The unit circle S^1 is a Lie group. There are two ways to see this. One is by considering S^1 in $\mathbb{C}^*$ with multiplication induced from $\mathbb{C}^*$ The other is by using the identification $S^1 = \mathbb{R}/\mathbb{Z}$. The set $\mathbb{Z}$ of integers is a normal subgroup of $\mathbb{R}$, and so $\mathbb{R}/\mathbb{Z}$ is a group, and since it is discrete, $\mathbb{R}/\mathbb{Z}$ is also a manifold. The smooth addition of $\mathbb{R}$ induces a smooth addition in $\mathbb{R}/\mathbb{Z}$.

(4) The product $G \times H$ of two Lie groups is itself a Lie group with the product manifold structure, and multiplication $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$.

(5) The *n-torus* $T^n = S^1 \times \quad \times S^1$ (n times) is a Lie group of dimension n.

(6) The *general linear group* $\mathrm{GL}_n\mathbb{R}$ of all invertible $n \times n$ real matrices has been mentioned in Section 1. It can also be identified with the set $\mathrm{Aut}(\mathbb{R}^n)$ of all invertible linear maps from $\mathbb{R}^n$ to itself. Similarly, we can define the sets $\mathrm{GL}_n\mathbb{C}$ and $\mathrm{GL}_n\mathbb{H}$.

The following examples of Lie groups are obtained as closed subgroups of the various general linear groups, so we need the following definitions.

Definition. (a) A *Lie subgroup* of a Lie group G is a Lie group H that is an abstract subgroup and an immersed submanifold of G.

(b) A *closed subgroup* of a Lie group G is an abstract subgroup and a closed subset of G.

Notice that for the case of a Lie subgroup, H need not have the induced topology. However, the following theorem is true [**War**]:

Theorem 1.4. *If H is a closed subgroup of a Lie group G, then H is a submanifold of G and hence a Lie subgroup of G. In particular, it has the induced topology.*

We can now give more examples of Lie groups that are defined by using functions on $\mathrm{M}_n\mathbb{R}$ such as the determinant, transpose and complex conjugate, hence are Lie groups by the previous theorem.

(7) The *special linear group* is $\mathrm{SL}_n\mathbb{R} = \{A \in \mathrm{GL}_n\mathbb{R}\colon \det A = 1\}$.

(8) The *orthogonal group* is the group $O(n) = \{A \in \mathrm{GL}_n\mathbb{R}\colon AA^t = I\}$. The condition $AA^t = I$ is equivalent to $A^{-1} = A^t$ and so $O(n) = f^{-1}(0)$, where $f\colon \mathrm{GL}_n\mathbb{R} \to \mathrm{M}_n\mathbb{R}$ with $f(A) = A^{-1} - A^t$.

(9) The *unitary group* is the group $U(n) = \{A \in \mathrm{GL}_n\mathbb{C}\colon A\bar{A}^t = I\}$.

(10) The *symplectic group* is $Sp(n) = \{A \in \mathrm{GL}_n\mathbb{H}\colon A\bar{A}^t = I\}$. (The conjugate of the quaternion $q = t + ix + jy + kz$ is $\bar{q} = t - ix - jy - kz$.) Sometimes it is more convenient to use the equivalent definition $Sp(n) = \{A \in U(2n)\colon A^tJ = JA^{-1}\}$, where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$.

(11) The *special orthogonal group* is the group $SO(n) = \{A \in O(n)\colon \det A = 1\}$.

(12) The *special unitary group* is the group $SU(n) = \{A \in U(n)\colon \det A = 1\}$.

Examples (6), (7), (10)-(12) are known as the *classical* groups.

The Lie groups $O(n), U(n)$ and $Sp(n)$ can also be defined as groups of linear isometries of $\mathbb{R}^n, \mathbb{C}^n$ and $\mathbb{H}^n$ respectively as follows: Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. We define an inner product $\langle\ \ \rangle$ on $\mathbb{K}^n$ by $\langle x, y\rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$, and let

$$O(n, \mathbb{K}) = \{A \in \mathrm{M}_n\mathbb{K}\colon \langle xA, yA\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{K}^n\}.$$

Then it is easy to ckeck that the set $O(n, \mathbb{K})$ is a group equal to $O(n), U(n)$ and $Sp(n)$, when $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ respectively.

Here are some simple group isomorphisms for certain matrix Lie groups. Let $S^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}$. Then

$$SO(1) \cong SU(1) \cong \{e\},\ O(1) \cong S^0,\ SO(2) \cong U(1) \cong S^1,\ SU(2) \cong S^3$$

We check the last two. For the first, let $z = x + iy$ be a complex number identified with the point (x, y) in $\mathbb{R}^2$. There is a one-to-one correspondence between complex numbers and certain 2×2 real matrices given by

$$z = x + iy \in \mathbb{C} = \mathbb{R}^2 \leftrightarrow A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in M_2\mathbb{R}.$$

Now, the ordinary scalar product on $\mathbb{R}^2$ defined by $\langle z_1, z_2 \rangle = x_1x_2 + y_1y_2$ ($z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$) can also be expressed as $\frac{1}{2}(z_1\bar{z}_2 + \bar{z}_1z_2)$. An easy computation shows that this scalar product corresponds to the scalar product $\langle A_1, A_2 \rangle = \frac{1}{2}\operatorname{tr}(A_1A_2^t)$ in the above matrix model of $\mathbb{R}^2$. Furthermore, the length of a complex number $|z|^2 = \langle z, z \rangle$ becomes $|A|^2 = \det A$. By definition, $S^1 = \{z = (x, y) \in \mathbb{C} \colon |z| = 1\}$, and in the matrix model of $\mathbb{R}^2$ this corresponds to the set of matrices $\{A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \ \det A = 1\}$. This set of matrices can also be described as $\{B \in M_2\mathbb{R} \colon BB^t = I, \det M = 1\}$, which is the group $SO(2)$. This correspondence is a group isomorphism, where the operation in both groups is multiplication.

We now come to the second isomorphism. Let $q = t + ix + jy + zk$ be a quaternion, which is identified with the point (t, x, y, z) in $\mathbb{R}^4$, or the point (v_1, v_2) in $\mathbb{C}^2$, where $v_1 = t + ix$ and $v_2 = y + iz$. There is a one-to-one correspondence between the quaternions and certain 2×2 complex matrices given by

$$q = t + ix + jy + zk \in \mathbb{H} = \mathbb{R}^4 \leftrightarrow A = \begin{pmatrix} v_1 & v_2 \\ -\bar{v}_2 & \bar{v}_1 \end{pmatrix} \in M_2\mathbb{C},$$

where $v_1 = t + ix$ and $v_2 = y + iz$. Via this correspondence the ordinary scalar product on R^4 becomes $\langle A_1, A_2 \rangle = \frac{1}{2}\operatorname{tr}(A_1\bar{A}_2^t)$, and, as before $|A|^2 = \det A$. By definition, $S^3 = \{q \in \mathbb{H} \colon |q| = 1\}$. This is a (non-commutative) group via quaternion multiplication. In the matrix model of $\mathbb{R}^4$, S^3 corresponds to the set of matrices $\{A = \begin{pmatrix} v_1 & v_2 \\ -\bar{v}_2 & \bar{v}_1 \end{pmatrix} \in M_2\mathbb{C}\colon \det A = 1\}$. This set can also be described as $\{B \in M_2\mathbb{C}\colon B\bar{B}^t = I, \det B = 1\}$, which is the group $SU(2)$. Hence the correspondence $S^3 \leftrightarrow SU(2)$ is a group isomorphism. Notice also, that $Sp(1)$ is isomorphic to S^3 We refer to [**Zu**] for a further discussion of the 3-sphere.

All the above group isomorphisms are manifold diffeomorphisms. A result of H. Hopf states that S^0, S^1, S^3 are the only spheres that admit a Lie group structure.

4. The tangent space of a Lie group - Lie algebras

Lie groups are non-linear objects and their study requires quite a lot of effort. On the other hand, one of the simplest algebraic objects is that of a real vector space. In this section we will see that it is possible to associate to every point of a Lie group G a real vector space, which is the tangent space of the Lie group at that point. By use of certain diffeomorphisms on the Lie group (left or right translations) we will see that it is enough to study the tangent space of a Lie group at its identity element e. The tangent space at that point is not only a vector space but, as we will see below, it is isomorphic to what is defined below to be the Lie algebra of the Lie group G. It is an object of special importance for the study of a Lie group and its geometry as we will see later on.

Let a be an element of a Lie group G. We define the maps

$$L_a\colon G \to G,\ \ L_a(g) = ag\ \ (\textit{left translation}),$$
$$R_a\colon G \to G,\ \ R_a(g) = ga\ \ (\textit{right translation}).$$

These maps are smooth, in fact they are diffeomorphisms since, for example, the inverse of L_a is $L_{a^{-1}}$. Furthermore, they can be used in order to get around in a Lie group. For instance, any $a \in G$ can be moved to e by $L_{a^{-1}}$. Finally, the induced map $(dL_{g^{-1}})_g\colon T_gG \to T_eG$

is a vector space isomorphism (similarly for the right translations). Hence we obtain the following:

Proposition 1.5. *Any Lie group G is parallelizable, i.e. $TG \cong G \times T_eG$.*

Proof. Let X_g be the value of a vector field X at a point $g \in G$. Then the map $X_g \mapsto (g, dL_{g^{-1}}(X_g))$ is the desired isomorphism. □

The following special class of vector fields on a Lie group will play an important role from now on.

Definition. A vector field X on a Lie group G is *left-invariant* if $X \circ L_a = dL_a(X)$ for all $a \in G$, or more explicitly $X_{ag} = (dL_a)_g(X_g)$ for all $a, g \in G$.

A left-invariant vector field has the important property that it is determined by its value at the identity element e of the Lie group, since $X_a = dL_a(X_e)$ for all $a \in G$. Also, since multiplication in G is smooth, so is a left-invariant vector field.

Let $\mathfrak{g}$ denote the set of all left-invariant vector fields on a Lie group G. The usual addition of vector fields and scalar multiplication by real numbers make $\mathfrak{g}$ a vector space. Furthermore, $\mathfrak{g}$ is closed under the bracket operation on vector fields. Indeed, let X, Y be two left-invariant vector fields on G, $a, p \in G$, and f a smooth function on G. Then we have

$$\begin{aligned} dL_a[X,Y]_p f &= [X,Y]_p(f \circ L_a) = X_p(Y(f \circ L_a)) - Y_p(X(f \circ L_a)) \\ &= X_p(dL_a Y)f - Y_p(dL_a X)f = X_p Y(f) - Y_p X(f) \\ &= (X_p Y - Y_p X)f = [X,Y]_p f, \end{aligned}$$

which shows that the bracket of two left-invariant vector fields is again a left-invariant vector field. Thus $\mathfrak{g}$ is a Lie algebra, called the *Lie algebra of* G. The dimension of this Lie algebra is equal to the dimension of G because of the following:

Proposition 1.6. *The function $X \mapsto X_e$ defines a linear isomorphism between the vector spaces $\mathfrak{g}$ and T_eG.*

Proof. The function is obviously linear, and it is one-to-one, since if $X_e = 0$, then $X_g = dL_g(X_e) = 0$ for all $g \in G$. The function is also onto: Let $v \in T_eG$ and define the vector field X^v by $X_g^v = (dL_g)_e(v)$ for all $g \in G$. Then X^v is left-invariant and $X_e^v = v$. □

Through this isomorphism we can define a Lie bracket on the tangent space T_eG by $[u, v] = [X^u, X^v]_e$.

Examples.

(1) The set $\mathrm{M}_n\mathbb{R}$ of all $n \times n$ real matrices is a Lie algebra if we set $[A, B] = AB - BA$.

(2) The Lie algebra of the general linear group $\mathrm{GL}_n\mathbb{R}$ is (canonically isomorphic) to $\mathrm{M}_n\mathbb{R}$, the set of all $n \times n$ real matrices. Indeed, recall that $\mathrm{GL}_n\mathbb{R}$ inherits its manifold structure as an open submanifold of $\mathrm{M}_n\mathbb{R}$. Hence we obtain the following canonical vector space isomorphisms:

$$\text{Lie algebra of } \mathrm{GL}_n\mathbb{R} \cong T_e(\mathrm{GL}_n\mathbb{R}) \cong T_e(\mathrm{M}_n\mathbb{R}) \cong \mathrm{M}_n\mathbb{R}$$

where e is the $n \times n$ identity matrix. The first isomorphism is obtained from Proposition 1.6, the second is the open submanifold identification, and the third one is the canonical vector space identification. By a straightforward coordinate calculation we see that brackets are also preserved. Similarly, the Lie algebras of $\mathrm{GL}_n\mathbb{C}$ and $\mathrm{GL}_n\mathbb{H}$ are $\mathrm{M}_n\mathbb{C}$ and $\mathrm{M}_n\mathbb{H}$ respectively.

(3) Let V be a vector space of dimension n. Let $\mathrm{End}(V)$ denote the set of all linear maps from V to itself (*endomorphisms of* V—this is diffeomorphic to $\mathrm{M}_n\mathbb{R}$), and let $\mathrm{Aut}(V)$ be the set of invertible linear maps (*automorphisms of* V—this is diffeomorphic to $\mathrm{GL}_n\mathbb{R}$). The set $\mathrm{End}(V)$ becomes a Lie algebra of dimension n^2 if we set $[f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1$. On the other hand, $\mathrm{Aut}(V)$ is a Lie group (it inherits a manifold structure as an open subset of $\mathrm{End}(V)$, and the group operation is the composition of maps), and $T_e(\mathrm{Aut}(V)) = \mathrm{End}(V)$. (Here, e denotes the identity transformation on V.)

(4) The previous examples extend to complex matrices. The Lie algebra of $\mathrm{GL}_n\mathbb{C}$ is $\mathrm{M}_n\mathbb{C}$.

5. One-parameter subgroups

Here we will describe a second characterization of the tangent space of a Lie group as the set of its one-parameter subgroups. This is also called the *infinitesimal* description of a Lie group and it is actually what Lie called an "infinitesimal group"

Definition. A *one-parameter subgroup* of a Lie group G is a smooth homomorphism $\phi\colon (\mathbb{R},+) \to G$.

Thus $\phi\colon \mathbb{R} \to G$ is a curve such that $\phi(s+t) = \phi(s)\phi(t)$, $\phi(o) = e$, and $\phi(-t) = \phi(t)^{-1}$.

Examples.

(1) The map $\phi(t) = e^t$ is a one-parameter subgroup of the additive Lie group $\mathbb{R}$.

(2) The map $\phi(t) = e^{it}$ is a one-parameter subgroup of the circle $S^1 = U(1)$.

(3) The map $\phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a one-parameter subgroup in $U(2)$.

(4) The map

$$\phi(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

is a one-parameter subgroup in $\mathrm{GL}_3\mathbb{R}$.

The central theorem here is the following:

Theorem 1.7. *The map $\phi \mapsto d\phi_0(1)$ defines a one-to-one correspondence between one-parameter subgroups of G and T_eG.*

Proof. Let $v \in T_eG$ and $X_g^v = (dL_g)_e(v)$ be (the value of) the corresponding left-invariant vector field. We need to find a smooth homomorphism $\phi_v\colon \mathbb{R} \to G$. By Proposition 1.2 let $\phi\colon I \to G$ be the unique integral curve of X^v such that $\phi(0) = e$ and $d\phi_t = X^v_{\phi(t)}$. This curve is a homomorphism because if we fix an $s \in I$ such that

$s+t \in I$ for all $t \in I$, then the curves $t \mapsto \phi(s+t)$ and $t \mapsto \phi(s)\phi(t)$ satisfy the previous equation (the second curve by the left-invariance of X^v), and take the common value $\phi(s)$ when $t = 0$. Thus by the uniqueness of the solution we obtain that

$$\phi(s+t) = \phi(s)\phi(t) \qquad (s, t \in I).$$

Now extend ϕ to all of $\mathbb{R}$ by defining $\phi_v(t) = \phi(\frac{t}{n})^n$ for suitably large n, and this is the desired homomorphism. The map $v \mapsto \phi_v$ is the inverse of $\phi \mapsto d\phi_0(1)$ and this completes the proof. $\square$

By using the identification of the tangent space T_eG with $\mathfrak{g}$, the set of all left-invariant vector fields in G, we obtain the following:

Corollary 1.8. *For each $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup $\phi_X : \mathbb{R} \to G$ such that $\phi'_X(0) = X$.*

Hence it is possible to organize the set of all one-parameter subgroups of a Lie group G into a single map $\mathfrak{g} \to G$ as follows:

Definition. The *exponential map* $\exp\colon \mathfrak{g} \to G$ is defined by $\exp(X) = \phi_X(1)$, where ϕ_X is the unique one-parameter subgroup of X.

Next, we will find the relation between ϕ_X and ϕ_{sX} $(s \in \mathbb{R})$. Consider the map $h(t) = \phi_X(st)$. This is a one-parameter subgroup with $h'(t) = s\phi'_X(st)$, so $h'(0) = s\phi'_X(0) = sX$. On the other hand, by Corollary 1.8 $\phi'_{sX}(0) = sX$, hence by uniqueness it follows that

$$\phi_X(st) = \phi_{sX}(t).$$

If we interchange the roles of s, t in the above relation and take $s = 1$ we also obtain that

$$\exp(tX) = \phi_{tX}(1) = \phi_X(t),$$

hence we obtain the following:

Corollary 1.9. *The curve $\gamma(t) = \exp(tX)$ $(X \in \mathfrak{g})$ is the unique homomorphism in G with $\gamma'(0) = X$. Also, since ϕ_X is a homomorphism, it follows that $\exp(s+t)X = \exp sX \cdot \exp tX$ and $(\exp tX)^{-1} = \exp(-tX)$.*

Now we compute the differential $(d\exp)_{\mathbf{o}}\colon \mathfrak{g} \to \mathfrak{g}$ of the exponential map at $\mathbf{o} \in \mathfrak{g}$. Take the curve $\alpha(t) = tX$ in $\mathfrak{g}$ with $\alpha(0) = \mathbf{o}$ and $\alpha'(0) = X \in \mathfrak{g}$. Then

$$(d\exp)_{\mathbf{o}}(X) = \frac{d}{dt}(\exp\circ\alpha)\bigg|_{t=0} = \frac{d}{dt}(\exp(tX))\bigg|_{t=0} = X,$$

so $(d\exp)_{\mathbf{o}}$ is the identity map. By applying the inverse mapping theorem we obtain:

Proposition 1.10. *There is a neighborhood of* $\mathbf{o} \in \mathfrak{g}$ *which is mapped diffeomorphically by* exp *onto a neighborhood of* $e \in G$.

If V is such a neighborhood of $\mathbf{o}$, then $\exp(V) = U$ is called a *normal neighborhood* of p.

Before we see various examples we will comment on the term "infinitesimal" group. If $\phi(t)$ is a one-parameter subgroup of G, then we can express its derivative as follows:

$$\phi'(t) = \lim_{h\to 0}\frac{1}{h}[\phi(t+h) - \phi(t)] = \lim_{h\to 0}\frac{1}{h}[(\phi(h) - e)\phi(t)] = A\phi(t),$$

where A is the limit as $h \to 0$ of $(\phi(h)-e)/h$. This limit exists because the group is a manifold whose coordinates are smooth functions. Now, if A is a matrix, we will see next that the matrix e^{At} is defined and the curve $\phi(t) = e^{At}$ is the (unique) solution of the above differential equation with the initial condition $\phi(0) = A$. The matrix A is called the *infinitesimal generator* of the subgroup $\phi(t)$. For example, if we take the one-parameter subgroup $\phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ of $U(2)$, then

$$\phi'(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

The infinitesimal generator of $\phi(t)$ given above is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the sense that $\phi(t) = e^{At}$ (matrix exponentiation).

Examples.

(1) Take $G = \mathrm{GL}_n\mathbb{R}$ with $\mathfrak{g} = \mathrm{M}_n\mathbb{R}$. In this example the term "exponential map" will be justified since it will coincide with the usual exponential map for matrices. Let A be an $n \times n$ real (or complex matrix). Define the $n \times n$ matrix

$$e^A = I + A + \frac{A^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

The above series converges in $\mathrm{M}_n\mathbb{R}$. In fact, define the norm $||A|| = \max_{1 \le i \le n} |A|_i$, where $|A|_i$ is the sum of the absolute values of the entries in the i^{th} row. Then $||AB|| \le ||A|| \ \ ||B||$, and $||A + B|| \le ||A|| + ||B||$. Thus

$$||e^A|| \le I + ||A|| + \frac{||A||^2}{2!} + \cdots = e^{||A||},$$

which shows that the series converges absolutely, so it converges (this norm gives the expected topology in $\mathrm{M}_n\mathbb{R}$). Now, the exponential map $\exp\colon \mathrm{M}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$ is given by $\exp(A) = e^A$. Indeed, let $A \in \mathrm{M}_n\mathbb{R}$. Then the map $\phi(t) = e^{tA}$ is a one-parameter subgroup in $\mathrm{GL}_n\mathbb{R}$ with $\phi'(0) = A$, so by Corollary 1.8 it is the exponential map $\exp\colon \mathrm{M}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$. As a consequence of this, we can obtain that for any $A, B \in \mathrm{M}_n\mathbb{R}$ the Lie bracket is given by

$$[A, B] = AB - BA.$$

(This is also true for $A, B \in \mathrm{M}_n\mathbb{C}$ or $\mathrm{M}_n\mathbb{H}$.)

Here is a sketch of the proof: Let X^A, X^B be the corresponding left-invariant vector fields to A, B. Then $[A, B] = [X^A, X^B]_e$. On the other hand, it is an exercise to show that for any two vector fields X, Y on a manifold M with corresponding flows α_t and β_t through a point p which is close to a fixed point $0 \in M$, the bracket is given by $[X, Y]_0 = \lim_{t \to 0} \gamma'(t)$, where

$$\gamma(t) = \beta_{-\sqrt{t}}(\alpha_{-\sqrt{t}}(\beta_{\sqrt{t}}(\alpha_{\sqrt{t}}(0)))).$$

Now, an integral curve for X^A through a point $g \in \mathrm{GL}_n\mathbb{R}$ is $t \mapsto g \exp tA$. So in this case

$$\gamma(t) = \exp(-\sqrt{t}A)\exp(-\sqrt{t}B)\exp(\sqrt{t}A)\exp(\sqrt{t}B),$$

and since we have matrix groups $\gamma(t) = e^{-\sqrt{t}A}e^{-\sqrt{t}B}e^{\sqrt{t}A}e^{\sqrt{t}B}$ Finally, the computation of the above limit gives that this is equal to $AB - BA$.

(2) The Lie algebra of the orthogonal group $O(n)$ is the set $\mathfrak{o}(n) = \{A \in \mathrm{M}_n\mathbb{R}\colon A^t = -A\}$ of all *skew-symmetric* real matrices. Matrices obeying the condition $A^t = -A$ vanish on the diagonal. Hence, $\dim O(n) = \frac{1}{2}n(n-1)$.

Indeed, we will show that $T_IO(n) = \mathfrak{o}(n)$, where I is the identity matrix. Let $\gamma(s)$ be a curve in $\mathrm{M}_n\mathbb{R}$ with $\gamma(0) = I$, that lies in $O(n)$, i.e., $\gamma(s)^t\gamma(s) = I$. Differentiating at $s = 0$ we obtain that $\gamma'(0)^t = -\gamma'(0)$, thus $T_IO(n) \subset \mathfrak{o}(n)$. Conversely, let $A \in \mathfrak{o}(n)$. Then $\gamma(s) = e^{sA}$ is a curve in $\mathrm{M}_n\mathbb{R}$ with $\gamma(0) = I$ and $\gamma(\mathbb{R}) \subset O(n)$ (since $(e^{sA})^t = (e^{sA})^{-1}$). Differentiating at $s = 0$ we obtain $\gamma'(0) = A \subset T_IO(n)$, so $\mathfrak{o}(n) \subset T_IO(n)$.

(3) Similarly, it can be shown that the Lie algebra of the unitary group $U(n)$ is the set $\mathfrak{u}(n) = \{A \in \mathrm{M}_n\mathbb{C}\colon \bar{A}^t = -A\}$ of all *skew-hermitian* complex matrices. The diagonal entries are all pure imaginary and $\dim U(n) = n^2$.

(4) The Lie algebra of $SU(n)$ is the set $\mathfrak{su}(n) = \{A \in \mathrm{M}_n\mathbb{C}\colon \bar{A}^t = -A \text{ and } \mathrm{tr}A = 0\}$. Here one uses the relation $\det(e^{tA}) = e^{t\mathrm{tr}A}$, which is obtained first for an upper triangular matrix and then, by using the Jordan canonical form, for any matrix A. We obtain that $\dim SU(n) = n^2 - 1$.

(5) Similarly, the Lie algebra of $\mathrm{SL}_n\mathbb{R}$ is $\mathfrak{sl}_n\mathbb{R} = \{A \in \mathrm{M}_n\mathbb{R}\colon \mathrm{tr}A = 0\}$. Its dimension is $n^2 - 1$.

(6) The Lie algebra of the special orthogonal group $SO(n)$ is the same as the Lie algebra of $O(n)$, i.e., $\mathfrak{so}(n) = \mathfrak{o}(n)$. Hence $\dim SO(n) = \frac{1}{2}n(n-1)$.

(7) The Lie algebra of the symplectic group $Sp(n)$ is the set $\mathfrak{sp}(n) = \{A \in \mathrm{M}_n\mathbb{H}\colon \bar{A}^t = -A\}$. Due to the appearance of the conjugation in the quaternions, it is more helpful to view this as isomorphic to $\{A \in \mathrm{M}_{2n}\mathbb{C}\colon \bar{A}^t = -A \text{ and } A^tJ + JA = 0\}$.

Concerning the topology of these groups, $SO(n), O(n), SU(n), U(n)$ and $Sp(n)$ are compact (they are closed and bounded subsets of corresponding general linear groups). The groups $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ are connected. The orthogonality condition for $O(n)$ implies that if $A \in O(n)$, then $\det A = \pm 1$, thus $O(n)$ has two connected components, one of which is $SO(n)$.

6. The Campbell-Baker-Hausdorff formula

The relation between $\exp X \exp Y$ and $\exp(X + Y)$ is given by the Campell-Baker-Hausdorff (CBH) formula. For the proof we refer to [**Fe**] or [**War**].

Theorem 1.11. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there exists a curve $t \mapsto Z(t)$ in $\mathfrak{g}$ such that*

$$\exp(tX)\exp(tY) = \exp(Z(t)),$$

where $Z(t)$ has a Taylor series expansion $Z(t) = \sum_{n=1}^{\infty} Z_n(X,Y)$, with $Z_1(X,Y) = [X,Y]$, $Z_2(X,Y) = \frac{1}{2}[X,Y]$, and $Z_3(X,Y) = \frac{1}{12}[[X,Y],Y] - \frac{1}{12}[[X,Y],X]$.

Since in most applications it is only the $Z_1(X,Y)$ term that is needed, the above formula is also written as

$$\exp(tX)\exp(tY) = \exp\{t(X+Y) + O(t^2)\},$$

where $O(t^2)$ is a $\mathfrak{g}$-valued smooth function of t such that $\frac{1}{t^2}O(t^2)$ is bounded at $t = 0$.

This theorem has several consequences. For example, Theorem 1.4 is obtained by using the CBH formula. Also, one can show that a Lie group is *abelian* (i.e., $xy = yx$ for all $x, y \in G$) if and only if $\mathfrak{g}$ is *commutative* (i.e., $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$). Also, if G is abelian, then $\exp X \exp Y = \exp(X+Y)$. Finally, we can find all abelian Lie groups:

Proposition 1.12. *Every connected abelian Lie group is of the form $T^k \times \mathbb{R}^{n-k}$, where $T = \mathbb{R}/\mathbb{Z}$. If G is in addition compact, then G is a torus.*

7. Lie's theorems

We will present various theorems that give the precise relation between Lie groups and their Lie algebras. They are not necessarily in the original form of Lie's formulation. For proofs and more details we refer to **[Ca-Se-Mc]**, **[Du-Ko]**, **[Hs]**. We will need the following definition.

Definition. Let $\mathfrak{g}$ be an (abstract) Lie algebra and $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$.

(a) $\mathfrak{h}$ is called a *Lie subalgebra* of $\mathfrak{g}$, if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.
(b) $\mathfrak{h}$ is called an *ideal* in $\mathfrak{g}$, if $[A, X] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $A \in \mathfrak{g}$.

The next proposition gives a first relation between Lie groups and Lie algebras.

Proposition 1.13. *Let $\phi\colon G \to H$ be a Lie group homomorphism. Then the map $d\phi_e\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism (i.e., a vector space homomorphism that preserves the Lie brackets in G and H). Furthermore,*

$$\phi(\exp X) = \exp(d\phi_e(X)).$$

We now list Lie's results:

Theorem 1.14. (1) *For any Lie algebra $\mathfrak{g}$ there is a Lie group G (not necessarily unique) whose Lie algebra is $\mathfrak{g}$.*

(2) *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, for each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup H of G which has $\mathfrak{h}$ as its Lie algebra. Furthermore, normal subgroups of G correspond to ideals in $\mathfrak{g}$.*

(3) *Let G_1, G_2 be Lie groups with corresponding Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Then if $\mathfrak{g}_1 \cong \mathfrak{g}_2$ (isomorphic as Lie algebras), then G_1 and G_2 are locally isomorphic. If the Lie groups G_1, G_2 are simply connected (i.e. their fundamental groups are trivial), then G_1 is isomorphic to G_2.*

Remarks. (a) Part (1) in Theorem 1.14 is a consequence of Ado's theorem which states that any finite-dimensional (abstract) real Lie algebra is isomorphic to a Lie subalgebra of the Lie algebra $\mathrm{GL}_n\mathbb{R}$ for sufficiently large n.

(b) Concerning part (2), if $j\colon H \to G$ is the immersion of H in G, then the required subalgebra is $dj_e(\mathfrak{h})$. Conversely, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the required subgroup is the one generated by $\{\exp tX\colon X \in \mathfrak{h}\}$. If we drop the connectedness condition, then the Lie subgroup is not unique. For example, the Lie groups $O(n)$ and $SO(n)$ have the same Lie algebra $\mathfrak{o}(n)$.

(c) We cannot drop the simply connectedness condition for G_1 in part (3) as, for example, $G_1 = S^1$ and $G_2 = \mathbb{R}$ have the same Lie algebras, but are not isomorphic.

(d) Part (3) can be restated in categorical language as follows: There is a one-to-one correspondence between (i) the category of connected, simply connected Lie groups and Lie group homomorphisms, and (ii) the category of Lie algebras and Lie algebra homomorphisms.

Chapter 2

Maximal Tori and the Classification Theorem

The aim of this chapter is to discuss the classification problem for a compact and connected Lie group. However, our main intention is not to give a complete treatment of the classification problem, but to present various fundamental concepts (such as the adjoint representation, and the maximal tori) to be used later on. We will also briefly present the structure theory of the complex semisimple Lie algebras. We will need this later on, when it will be more convenient to treat various geometrical problems about Lie groups or homogeneous spaces at the Lie algebra level.

As we will see in a more precise statement in Theorem 2.17, it is a remarkable fact that the compact and simple groups come in three families ($SU(n)$, $SO(n)$, $Sp(n)$), plus five exceptional groups. A general guideline in mathematics when one needs to approach a classification problem is to develop useful invariants for the objects to be studied. For example, for the case of manifolds such invariants are the dimension, the homotopy groups, and the homology groups. For Lie groups we already know one invariant: the dimension. Since our objects are also groups, we have another invariant that can be considered: the center. For instance, if the group is abelian, then it coincides with its center. The first concept to be developed, the adjoint representation of a Lie group is, among other things, a measure of the non-commutativity of the group. We will start with some introductory concepts on representations.

1. Representation theory: elementary concepts

There are various reasons for looking at representations. For example, a representation is a useful tool for understanding the group and its possible invariants. Also, as Lie groups are often the symmetry groups of spaces of functions (such as solutions of differential equations), knowing the ways in which a group can act helps to understand these spaces.

Definition. A (finite-dimensional) *representation* of a Lie group G is a homomorphism $\phi\colon G \to \mathrm{Aut}(V)$, where V is a (finite-dimensional) vector space. The dimension of the representation is the dimension of the vector space V

Denote the representation of G in V by (G, V) or simply by V The map ϕ is required to be continuous.

If (G, V) is a representation of G and $g \in G$, $v \in V$, then this defines an *action* $\Phi\colon G \times V \to G$ of G on V as follows: $\Phi(g, v) = \phi(g)(v)$. Indeed, if we denote $\Phi(g, v)$ by $g \cdot v$, then we easily obtain that $e \cdot v = v$ and $g_1 \; (g_2 \cdot v) = (g_1 g_2 \cdot v)$, for all $g_1, g_2 \in G$, $v \in V$ For this reason a representation (G, V) is also refered to as a *G-space*, and we may use both notations $g \cdot v$ and $\phi(g)(v)$.

If the space V is a real (respectively complex, or quaternionic) vector space and, if for all $g \in G$, the maps $\Phi(g)\colon V \to V \; v \mapsto \Phi(g, v)$ are linear, then the corresponding representation is called *real* (respectively *complex*, or *quaternionic*).

Definition. Let (G, V) be a representation. A subspace U of V is called *invariant* or G-invariant if $g \cdot U \subset U$ for all $g \in G$.

A representation (G, V) has always at least two invariant subspaces, namely $\{0\}$ and V The first is called the *trivial* subspace.

Definition. A representation is called *irreducible* if the only invariant subspaces are $\{0\}$ and V

Definition. Two representations $\phi_1 \colon G \to \mathrm{Aut}(V_1)$ and $\phi_2 \colon G \to \mathrm{Aut}(V_2)$ are said to be *equivalent* (denoted by $\phi_1 \cong \phi_2$) if V_1 and V_2 are G-isomorphic, i.e., there exists a linear isomorphism $A \colon V_1 \to V_2$ such that $A(\phi_1(g)(v)) = \phi_2(g)(A(v))$, for all $g \in G$ and $v \in V_1$. In shorthand $A\phi_1 = \phi_2 A$.

This means that the following diagram is commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi_1(g)} & V_1 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ V_2 & \xrightarrow{\phi_2(g)} & V_2 \end{array}$$

Given two representations (G, V) and (G, W) of G, we can define the following representations on the corresponding vector spaces with the obvious notations:

(a) Dual space: V^* with $\langle v, g \cdot v' \rangle = \langle g^{-1} \cdot v, v' \rangle$ for all $v \in V$, $v' \in V^*$

(b) Direct sum: $V \oplus W$ with $g \cdot (x, y) = (gx, gy)$.

(c) Tensor product: $V \otimes W$ with $g \cdot (x \otimes y) = g \cdot x \otimes g \cdot y$.

(d) The set $\mathrm{Hom}(V, W)$ of all homomorphisms from V to W, with $(g \cdot A)x = gA(g^{-1} \cdot x)$, for all $A \in \mathrm{Hom}(V, W)$.

(e) If V is a complex vector space, then we may define the conjugate space $\overline{V}$ that has the same addition as V but scalar multiplication defined by $\mathbb{C} \times V \to V$ $(z, v) \mapsto \bar{z}v$. Then $(G, \overline{V})$ is a representation of G. If $(\ ,\ )$ is a ϕ-invariant inner product on V, then the map $v \mapsto (\ , v)$ gives an isomorphism $\overline{V} \cong V^*$

We can also define representations on other algebraic objects such as wedge products $\wedge^k(V)$, or symmetric products $S^k(V)$.

Below we list a few canonical isomorphisms:

$$\begin{aligned} U\otimes(V\oplus W) &\cong (U\otimes V)\oplus(U\otimes W), \\ (V\otimes W)^* &\cong V^*\otimes W^*, \\ \mathrm{Hom}(V,W) &\cong V^*\otimes W, \\ \wedge^k(V\oplus W) &\cong \oplus_{i=0}^k(\wedge^i(V)\otimes\wedge^{k-i}(W)). \end{aligned}$$

Definition. Let (G,V) and (G,W) be two representations. A map $f\colon V\to W$ is called *G-equivariant* if $f(g\ \ v)=g\ \ f(v)$ for all $g\in G,\ v\in V$

Theorem 2.1 (Schur's Lemma – first version). *Let V and W be two irreducible representations of G, and let $f\colon V\to W$ be a G-equivariant map. Then either f is invertible or $f=0$.*

Proof. Clearly $\ker f\subset V$ and $\mathrm{Im}\ f\subset W$ are invariant subspaces. By the irreducibility assumption it follows that $\ker f$ is either $\{0\}$ or V, and $\mathrm{Im}\ f$ is either $\{0\}$ or W The only possibilities are either $\ker f=\{0\}$ and $\mathrm{Im}\ f=W$, i.e., f is invertible, or $\ker f=V$ and $\mathrm{Im}\ f=\{0\}$, i.e., $f=0$. □

Theorem 2.2 (Schur's Lemma – second version). *If V is an irreducible complex representation, and $f\in\mathrm{Hom}(V,V)$ is a G-equivariant map, then $f=\mathrm{c}\ \mathrm{Id}$ (identity map) for some $\mathrm{c}\in\mathbb{C}$.*

Proof. By the fundamental theorem of algebra, f has an eigenvalue, say $\mathrm{c}\in\mathbb{C}$. Then $f-\mathrm{c}\ \mathrm{Id}$ is a G-equivariant map of an irreducuble representation which is not invertible, and hence must be the zero map, i.e., $f=\mathrm{c}\ \mathrm{Id}$. □

Corollary 2.3. *If G is abelian, then any complex irreducible representation is one-dimensional.*

Proof. Let $\phi\colon G\to\mathrm{Aut}(V)$ be a complex irreducible representation. Since G is abelian, the map $\phi(g)$ is a G-equivariant self-map of V; therefore, $\phi(g)=\mathrm{c}(g)\ \mathrm{Id}$ for some complex scalar $\mathrm{c}(g)$. Since g is an arbitrary element of G, $\mathrm{Im}\ \phi\subset\mathbb{C}^*\ \mathrm{Id}$, so any subspace of V is G-invariant; therefore, it can be irreducible only when $\dim V=1$. □

Irreducible representations are the building blocks of any representation. To formulate this more precisely, we need the following theorem, whose proof requires the existence of a G-invariant inner product on V In order to do so, we first need the existence of a Haar integral on a compact Lie group G (see for example [**Si**]).

Theorem 2.4. *Let G be a compact Lie group (in fact any compact topological group), and $C(G)$ the set of all continuous real-valued functions on G. Then there exists a unique function $I\colon C(G) \to \mathbb{R}$ such that*

(a) $I(\mathbf{1}) = 1$,
(b) *I is positive*[1] *and linear,*
(b) *I is invariant, i.e., $I(f) = I(f \circ L_g) = I(R_g \circ f)$ for all $g \in G$.*

The number $I(f)$ is denoted by $\int_G f(g)dg$ and is called a *Haar integral* on G. It is usually realized by some form of integration on G.

Theorem 2.5. *Let $\phi\colon G \to \mathrm{Aut}(V)$ be a representation of a compact group G. Then there exists a G-invariant inner product $(\ ,\ )$ on V, i.e. $(g \cdot u, g \cdot v) = (u, v)$ for all $u, v \in G$ and $g \in G$.*

Proof. Take an inner product $\langle\ ,\ \rangle$ on V Then define

$$(u, v) = \int_G \langle \phi(g)u, \phi(g)v \rangle dg$$

for all $u, v \in V$ $\square$

A real (resp. complex) representation with a G-invariant inner (resp. Hermitian) product is called an *orthogonal* (resp. *unitary*) representation.

Theorem 2.6. *Any finite-dimensional representation of a compact group is a direct sum of irreducible representations.*

Proof. Let $\phi\colon G \to \mathrm{Aut}(V)$ be a representation. Then we proceed by induction on $\dim V$ If $\dim V = 1$ the result is true trivially. Suppose we have the result for all representations of dimension $< n$. Let U

[1] We say I is positive if $I(f) \geq 0$ for $f \geq 0$.

be a non-trivial invariant subspace of dimension n. If U is irreducible we stop. If not, let $U^\perp = \{v \in V\colon (v,u) = 0 \text{ for all } u \in U\}$ (by using an inner product in V). Now $U^\perp$ is an invariant subspace since

$$(\phi(g)v, u) = (v, \phi(g^{-1})u) = 0$$

for all $g \in G$, $v \in U^\perp$, $u \in U$, and $V = U \oplus U^\perp$ with dimensions of U and $U^\perp$ less than n. Also, if we define representations ϕ_1 and ϕ_2 of G on the subspaces U and $U^\perp$ by restriction of ϕ (these representations are denoted by $\phi \restriction U$ and $\phi \restriction U^\perp$ respectively), then $\phi \cong \phi_1 \oplus \phi_2$ (straightforward use of the definitions). By induction, each of U and $U^\perp$ is a direct sum of irreducibles, and so is V □

When we say that a representation $\phi\colon G \to \mathrm{Aut}(V)$ is a direct sum of irreducibles, we will mean that $V = V_1 \oplus \cdots \oplus V_k$ (V_i a subspace of V), the restricted (sub)representations $\phi_i\colon G \to \mathrm{Aut}(V_i)$ are irreducibles, and $\phi \cong \phi_1 \oplus \cdots \oplus \phi_k$ (equivalent representations).

2. The adjoint representation

An *automorphism* of a Lie group G is a map $\phi\colon G \to G$ that is a diffeomorphism and a group isomorphism. Let G be a Lie group and $x \in G$. Then the map $I_x\colon G \to G$ sending each g to xgx^{-1} is a homomorphism and, since $I_x = R_{x^{-1}} \circ L_x$, is a diffeomorphism, it is called an *inner automorphism* of G.

Definition. The *adjoint representation* of G is the homomorphism $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ given by $\mathrm{Ad}(g) = (dI_g)_e$.

This is a homomorphism since $I_{xy} = I_x \circ I_y$ implies that $\mathrm{Ad}_{xy} = \mathrm{Ad}_x \circ \mathrm{Ad}_y$ (we take differentials). It is also smooth (see [**War**]). We will see soon that, for familiar groups, the adjoint representation is a very familiar representation.

By taking the derivative of Ad we obtain a representation of $\mathfrak{g}$.

Definition. The *adjoint representation* of $\mathfrak{g}$ is the homomorphism $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(X) = (d\mathrm{Ad})_e(X)$.

Let $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$ and $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$ denote the *centers* of G and $\mathfrak{g}$ respectively. Then a first consequence is the following:

Proposition 2.7. *Let G be a connected Lie group. Then* $\ker \mathrm{Ad} = Z(G)$ *and* $\ker \mathrm{ad} = Z(\mathfrak{g})$. *Furthermore, the Lie algebra of $Z(G)$ is $Z(\mathfrak{g})$.*

A second consequence is the following important theorem:

Theorem 2.8. *The adjoint representation of $\mathfrak{g}$ satisfies* $\mathrm{ad}(X)Y = [X,Y]$ *for all $X, Y \in \mathfrak{g}$.*

Proof. By definition $\mathrm{Ad}(g)Y = dI_g(Y) = dR_{g^{-1}}dL_g(Y) = dR_{g^{-1}}Y$, for all $g \in G$ and $Y \in \mathfrak{g}$. Let $x_t = \exp(tX)$ be the flow of $X \in \mathfrak{g}$. Since X is left-invariant, $L_y \circ x_t = x_t \circ L_y$ for all $y \in G$, which gives that

$$x_t(y) = x_t(L_y(e)) = L_y(x_t(e)) = yx_t(e) = R_{x_t(e)}(y),$$

and therefore $dx_t = dR_{x_t(e)}$. Now we use Proposition 1.3 to compute:

$$\begin{aligned}
[X,Y] &= \lim_{t\to 0} \frac{1}{t}(Y - dx_t(Y)) = -\lim_{t\to 0} \frac{1}{t}(dR_{x_t(e)}(Y) - Y) \\
&= -\lim_{t\to 0} \frac{1}{t}(\mathrm{Ad}(x_t^{-1}(e))Y - Y) = \lim_{t\to 0} \frac{1}{t}(\mathrm{Ad}(x_t(e))Y - Y) \\
&= \mathrm{ad}(X)Y. \quad \square
\end{aligned}$$

This theorem shows that the bracket operation in $\mathfrak{g}$ measures the failure of G to be commutative. Indeed, if G is abelian, then $I_g = Id$, hence $\mathrm{Ad}_g = Id$ for all $g \in G$. Thus, by the proof of the above theorem $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$. A Lie algebra that satisfies this property is called *abelian.* The converse is true if G is connected (see [**War**]).

For the case of a *matrix group* (that is a subgroup of a general linear group), the adjoint representation has a simple expression:

Proposition 2.9. *If G is a matrix group, then* $\mathrm{Ad}(g)X = gXg^{-1}$ *for all $g \in G, X \in \mathfrak{g}$ (the multiplication being multiplication of matrices).*

Proof. Let $t \mapsto \exp(tX)$ be the one-parameter subgroup of X whose derivative at $t = 0$ is X. Since G is a matrix group, the exponential map is given by the ordinary exponentiation of matrices, and thus we have:

$$\begin{aligned}\mathrm{Ad}(g)X = (dI_g)_e(X) &= \frac{d}{dt} I_g(\exp tX)\bigg|_{t=0} = \frac{d}{dt} g(\exp tX)g^{-1}\bigg|_{t=0} \\ &= g\,\frac{d}{dt} e^{tX}\bigg|_{t=0} g^{-1} = gXg^{-1}. \quad \square\end{aligned}$$

A note on complexification. If V is a vector space over $\mathbb{R}$, then we can define the vector space $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ (or simply $V \otimes \mathbb{C}$), whose dimension over $\mathbb{C}$ equals the dimension of V over $\mathbb{R}$. We can formally think of $V \otimes_{\mathbb{R}} \mathbb{C}$ as the set

$$\{X + iY \colon X, Y \in V,\ \ i = \sqrt{-1}\}.$$

If $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, then the *complexification* of $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ (or sometimes written with the notation $\mathfrak{g} + i\mathfrak{g}$), with Lie bracket operation given by

$$[U + iV, X + iY] = [U, X] - [V, Y] + i([V, X] + [U, Y]).$$

If $T \colon V \to W$ is a linear map of vector spaces over $\mathbb{R}$, then we can define the *extension* $\overline{T} = T \otimes Id \colon V \otimes \mathbb{C} \to W \otimes \mathbb{C}$ of T by complex linearity, that is $\overline{T}(\sum v_i \otimes z_i) = \sum T(v_i) \otimes z_i$.

Now, if $\phi \colon G \to \mathrm{Aut}(V)$ is a representation of a Lie group G, we combine the previous concepts to define the *complexified* representation $\phi \otimes \mathbb{C} \colon G \to \mathrm{Aut}(V^{\mathbb{C}})$.

Examples.

(1) Let $G = SU(2)$ with Lie algebra $\mathfrak{su}(2)$ consisting of matrices of the form $\begin{pmatrix} is & z \\ -\bar{z} & -is \end{pmatrix}$. We will compute the adjoint representation $\mathrm{Ad} \colon SU(2) \to \mathrm{Aut}(\mathfrak{su}(2))$. Let $A = \begin{pmatrix} x + iy & u + iv \\ -u + iv & x - iy \end{pmatrix} \in SU(2)$. We know that $\mathrm{Ad}(A)$ is a non-singular linear transformation on $\mathfrak{su}(2)$

given by $\mathrm{Ad}(A)B = ABA^{-1}$. To find this transformation (actually the matrix that corresponds to this transformation), we pick a basis

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for $\mathfrak{su}(2)$, and a calculation on the first of these basis elements gives that

$$\begin{aligned}
&\mathrm{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\
&= \begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} x-iy & -u-iv \\ u-iv & x+iy \end{pmatrix} \\
&= \begin{pmatrix} ix^2+iy^2-iu^2-iv^2 & -2ixu+2uy+2xv+2ivy \\ 2iux+2xv-2uy+2ivy & iu^2+iv^2-ix^2-iy^2 \end{pmatrix}
\end{aligned}$$

By similar computations on the other basis elements we obtain that

$$\begin{aligned}
&\mathrm{Ad}\begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix} \\
&= \begin{pmatrix} x^2+y^2-u^2-v^2 & -2xv+2uy & 2xu+2yv \\ 2uy+2xv & x^2-y^2+u^2-v^2 & -2xy+2uv \\ -2xu+2yv & 2xy+2uv & x^2-y^2-u^2+v^2 \end{pmatrix}
\end{aligned}$$

Notice that this is a 3×3 matrix, which agrees with the dimension of the representation which is $\dim \mathfrak{su}(2) = 3$. From this example it is evident that the computation of the adjoint representation for $SU(n)$ is complicated in general. Hence it is often best to leave the adjoint representation as gXg^{-1}. However, by using some more advanced representation theory it is possible to express the adjoint representation of groups such as $SU(n)$ or $SO(n)$, as shown in the next example.

(2) Define the *standard representations* of $\mathrm{GL}_n\mathbb{R}$, $O(n)$ and $SO(n)$ on $\mathrm{M}_{n\times 1}(\mathbb{R}) \cong \mathbb{R}^n$, in which elements of these Lie groups operate by matrix multiplication on $\mathbb{R}^n$, i.e. $\phi(g)v = gv$. (Recall that a representation of G defines an action on the vector space V and vice-versa). Similarly, we can define the standard representations of the groups $\mathrm{GL}_n\mathbb{C}, SU(n)$ and $U(n)$ on $\mathbb{C}^n$. A representation is called *trivial* if each group element acts as the identity transformation. It is denoted by 1.

There is some notation used for the standard representations: denote by $\tilde{\lambda}_n$ the standard representation of $\mathrm{GL}_n\mathbb{R}$, and by λ_n the standard representation of $SO(n)$. Actually, $\lambda_n = \tilde{\lambda}_n\Big|_{SO(n)}$ Often the same symbol is also used for the standard representation of $O(n)$. Similarly, we denote by $\tilde{\mu}_n$ the standard representation of $\mathrm{GL}_n\mathbb{C}$ and by μ_n the standard representation of $SU(n)$ (or $U(n)$). Finally, let ν_n denote the standard representation of $Sp(n)$. Then the adjoint representations of these groups are equivalent to the following representations:

$$\begin{aligned}
&\mathrm{Ad}^{\mathrm{Gl}_n\mathbb{R}} = \tilde{\lambda}_n \otimes_{\mathbb{R}} \tilde{\lambda}_n^*, \\
&\mathrm{Ad}^{\mathrm{Gl}_n\mathbb{C}} = \tilde{\mu}_n \otimes_{\mathbb{C}} \tilde{\mu}_n^*, \\
&\mathrm{Ad}^{SO(n)} = \wedge^2\lambda_n \qquad \text{(similarly for } O(n)), \\
&\mathrm{Ad}^{U(n)} \otimes \mathbb{C} = \mu_n \otimes_{\mathbb{C}} \mu_n^* = \mu_n \otimes_{\mathbb{C}} \bar{\mu}_n, \\
&\mathrm{Ad}^{SU(n)} \otimes \mathbb{C} = \mu_n \otimes_{\mathbb{C}} \bar{\mu}_n - 1, \\
&\mathrm{Ad}^{Sp(n)} \otimes \mathbb{C} = S^2\nu_n.
\end{aligned}$$

Here S^2 and Λ^2 denote the second symmetric and exterior power respectively.

3. The Killing form

We have seen that for any representation (G, V) of a compact Lie group G, there exists a G-invariant inner product on V In particular, this happens for the adjoint representation of $(G, \mathfrak{g})$. We will now introduce an explicit inner product on $\mathfrak{g}$.

Definition. The *Killing form*[2] of a Lie algebra $\mathfrak{g}$ is the function $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(X,Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$.

The following proposition includes some of the properties of the Killing form.

Proposition 2.10. *The Killing form has the following properties:*

(a) *It is a symmetric bilinear form on* $\mathfrak{g}$.

[2]Named after Wilhelm Killing

(b) *If* $\mathfrak{g}$ *is the Lie algebra of* G, *then* B *is* Ad-*invariant, that is,* $B(X,Y) = B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y)$ *for all* $g \in G$ *and* $X, Y \in \mathfrak{g}$. *In other words, each* $\mathrm{Ad}(g), g \in G$ *is* B*-orthogonal.*

(c) *Each* $\mathrm{ad}(Z)$ *is skew-symmetric with respect to* B, *that is,*

$$B(\mathrm{ad}(Z)X, Y) = -B(X, \mathrm{ad}(Z)Y) \text{ or } B([X,Z],Y) = B(X,[Z,Y]).$$

Proof. (a) Bilinearity follows from the linearity of $X \mapsto \mathrm{ad}(X)$ and the linearity of the trace. Symmetry follows from $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) If $\sigma \colon \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ (i.e. a linear isomorphism with $\sigma[X,Y] = [\sigma X, \sigma Y]$), then $\mathrm{ad}(\sigma X) \circ \sigma = \sigma \circ \mathrm{ad}(X)$, or $\mathrm{ad}(\sigma X) = \sigma \circ \mathrm{ad}(X) \circ \sigma^{-1}$. Take $\sigma = \mathrm{Ad}(g)$ and compute:

$$\begin{aligned} B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) &= \mathrm{tr}(\mathrm{ad}(\mathrm{Ad}(g)X) \circ \mathrm{ad}(\mathrm{Ad}(g)Y)) \\ &= \mathrm{tr}(\mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1} \circ \mathrm{Ad}(g) \circ \mathrm{ad}(Y) \circ \mathrm{Ad}(g)^{-1}) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = B(X,Y). \end{aligned}$$

(c) We use the Jacobi identity twice and obtain:

$$\begin{aligned} [Z,[X,[Y,W]]] &= [[Z,X],[Y,W]] + [X,[Z,[Y,W]]] \\ &= [[Z,X],[Y,W]] + [X,[[Z,Y],W]] + [X,[Y,[Z,W]]]. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{ad}(Z) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) &= \mathrm{ad}(\mathrm{ad}(Z)X) \circ \mathrm{ad}(Y) \\ &\quad + \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)Y) \\ &\quad + \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z), \end{aligned}$$

or

$$[\mathrm{ad}(Z), \mathrm{ad}(X) \circ \mathrm{ad}(Y)] = \mathrm{ad}(\mathrm{ad}(Z)X) \circ \mathrm{ad}(Y) + \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)Y).$$

Since $\mathrm{tr}([A,B]) = 0$, we finally obtain that

$$B(\mathrm{ad}(Z)X, Y) + B(X, \mathrm{ad}(Z)Y) = 0. \quad \square$$

The Killing form of a Lie group G is understood to be the Killing form of its Lie algebra $\mathfrak{g}$.

Definition. A Lie group G is called *semisimple* if its Killing form is non-degenerate.

Historically the above definition is known as Cartan's criterion for semisimplicity. In that context, a semisimple Lie algebra is one that has no proper solvable ideas, i.e. whose radical is zero. To make these abstract algebraic definitions more concrete, simply think of a semisimple Lie algebra $\mathfrak{g}$ as one that has no proper subspaces $\mathfrak{h}$ with $[X,Y]=0$ if $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

Proposition 2.11. *If G is semisimple, then $Z(\mathfrak{g})=0$,*

Proof. Let $X \in Z(\mathfrak{g})$. Then $[X,Y]=0$ for all $Y \in \mathfrak{g}$, thus $\mathrm{ad}(X)$ is the zero operator, which gives that $B(X,X) = \mathrm{tr}(\mathrm{ad}X \mathrm{ad}X) = 0$. Since G is semisimple, $X = 0$. □

Corollary 2.12. *The center of a semisimple Lie group is discrete.*

The next theorem is important because it demonstrates for which Lie groups the Killing form defines an inner product.

Theorem 2.13. *If G is a compact semisimple Lie group, then its Killing form is negative definite.*

Proof. By Theorem 2.5, since G is compact, there is an Ad-invariant inner product on $\mathfrak{g}$, so $\mathrm{Ad}(g)$ is an orthogonal transformation of $\mathfrak{g}$. By a proof similar to (c) of Proposition 2.10, each $\mathrm{ad}(X)$ is skew-symmetric, so let $\mathrm{ad}(X) = (a_{ij})$ relative to an orthonormal basis of $\mathfrak{g}$. Then

$$B(X,X) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X)) = \sum_i \sum_j a_{ij} a_{ji} = -\sum_{i,j} a_{ij}^2 \leq 0.$$

Since G is semisimple, B is nondegenerate, so the above sum is strictly less than zero. □

The converse of this theorem is also true but harder to prove (cf. [**Fe**], [**He**]):

Theorem 2.14. *If G is a connected Lie group and B is negative definite on $\mathfrak{g}$, then G is compact and semisimple.*

Examples.

(1) We compute the Killing form of $SU(2)$. We first observe that, as we will prove in the next section, it suffices to compute the Killing form at certain simpler elements of the Lie algebra $\mathfrak{su}(2)$ (in this example at the diagonal elements). We use the basis for $\mathfrak{su}(2)$ used in example (1) of the previous section, and we compute that

$$\operatorname{ad}\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\theta \\ 0 & 2\theta & 0 \end{pmatrix}$$

Then a simple calculation using this basis gives that if $X = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}$ and $Y = \begin{pmatrix} i\phi & 0 \\ 0 & -i\phi \end{pmatrix}$, then

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) = -8\theta\phi = 4\operatorname{tr} XY.$$

(2) The Killing form of $U(2)$. We use the following basis for $\mathfrak{u}(2)$:

$$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

If $X = \begin{pmatrix} i\theta_1 & 0 \\ 0 & i\theta_2 \end{pmatrix}$ and $Y = \begin{pmatrix} i\phi_1 & 0 \\ 0 & i\phi_2 \end{pmatrix}$, then

$$\begin{aligned} B(X,Y) &= \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) = 4(\theta_1\phi_1 + \theta_2\phi_2) - 2(\theta_1+\theta_2)(\phi_1+\phi_2) \\ &= 4\operatorname{tr} XY - 2\operatorname{tr} X \operatorname{tr} Y. \end{aligned}$$

Notice that if $\theta_1 = \theta_2 = \phi_1 = \phi_2 = 1$, then $B(X,Y) = 0$, so $U(2)$ is not semisimple.

(3) The Killing form of $SO(3)$. Consider the basis for $\mathfrak{so}(3)$ that consists of the 3×3 matrices E_{12}, E_{13}, E_{23} that have 1 in the (i,j) entry, -1 in the (j,i) entry, and 0 elsewhere ($1 \leq i < j \leq 3$). As observed above, it suffices to compute the Killing form for the matrices

$X = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \phi & 0 \\ -\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ A computation gives that

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) = -2\theta\phi = \operatorname{tr} XY.$$

(4) The examples above can be generalized as follows:

$$\begin{aligned} U(n)&: & B(X,Y) &= 2n \operatorname{tr} XY - 2 \operatorname{tr} X \operatorname{tr} Y, \\ SU(n)&: & B(X,Y) &= 2n \operatorname{tr} XY, \\ SO(n)&: & B(X,Y) &= (n-2) \operatorname{tr} XY, \\ Sp(n)&: & B(X,Y) &= 2(n+1) \operatorname{tr} XY. \end{aligned}$$

4. Maximal tori

The key for the classification of compact and connected Lie groups are the maximal tori. The circle group S^1 is the only one-dimensional compact connected Lie group, and products of several copies of S^1 are the only commutative, compact and connected Lie groups. Such a product is called a torus.

Definition. A *torus* in a Lie group G is a Lie subgroup that is isomorphic to a product $S^1 \times \quad \times S^1$. A torus T is a *maximal* torus in G if for any torus S in G with $T \subset S \subset G$ for a torus S, then $T = S$.

Examples.

(1) The group of unit complex numbers S^1 is a maximal torus in the group of unit quaternions S^3

(2) The set $T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$ is a maximal torus in $SU(2)$.

(3) The set

$$T = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong SO(2)$$

is a maximal torus in $SO(3)$.

Remarks.

(1) Any torus is contained in a maximal torus. Indeed, if $T \subset T_1 \subset T_2 \subset \cdots$ is an increasing sequence of tori in G, then for their corresponding Lie algebras we have that $\mathfrak{t} \subset \mathfrak{t}_1 \subset \mathfrak{t}_2 \subset \cdots$. This is an increasing sequence of finite-dimensional vector spaces in $\mathfrak{g}$, hence it must be finite.

(2) If G is compact, then any maximal torus T is a maximal connected abelian subgroup of G. Indeed, if $T \subset A$ with A connected and abelian, then $T \subset \bar{A}$ (the closure of A), which is compact, since G is compact. But a compact, connected abelian group is a torus, and T is maximal, so $T = \bar{A}$.

(3) If T is a connected Lie subgroup of a compact Lie group G whose Lie algebra is a maximal abelian subalgebra of $\mathfrak{g}$, then T is a maximal torus in G.

The main result here is the following:

Theorem 2.15. *Let G be a compact and connected Lie group. Then:*

(a) *Any element in G is contained in some maximal torus.*

(b) *Any two maximal tori are conjugate. That is, if T_1, T_2 are maximal tori in G, then there exists an element $g \in G$ such that $gT_1g^{-1} = T_2$.*

From (b) in the above theorem, we see that every maximal torus has the same dimension. So this is an invariant for a compact and connected Lie group. Hence we can define:

Definition. The *rank* of a compact and connected Lie group is the dimension of a maximal torus.

For the proof of the above theorem as well as for its various consequences that we list in the next proposition, we refer to several sources (e.g.—from the most elementary to more advanced—[**Fe**], [**Si**], [**Du-Ko**], [**He**]). References [**Fe**] and partly [**Si**] provide a geometrical proof, which can be read after one reads Chapter 3.

Proposition 2.16. *Let G be a compact and connected Lie group with Lie algebra $\mathfrak{g}$. Then:*

(1) The exponential map is onto.

(2) There is a one-to-one correspondence between maximal tori T in G and maximal abelian subspaces $\mathfrak{h}$ in $\mathfrak{g}$. This is given by $T \leftrightarrow \mathfrak{h} = \exp \mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of T

(3) If T is a maximal torus in G with Lie algebra $\mathfrak{t}$, then $G = \cup_{g \in G} gTg^{-1}$ and $\mathfrak{g} = \cup_{g \in G} \mathrm{Ad}(g)\mathfrak{t}$.

(4) The center of G is equal to $\cap_{\substack{\text{maximal}\\ \text{tori}}} T$

(5) If S is a subset of G, we define the centralizer of S to be the set $C(S) = \{g \in G\colon gx = xg \text{ for all } x \in S\}$. Then, if T is a maximal torus in G, then $C(T) = T$

(6) Maximal tori are also maximal abelian subgroups.

(7) For any $X \in \mathfrak{g}$, the closure of $\{\exp(tX)\}$ is a compact abelian subgroup of G, and so a torus.

Examples.

(1) A maximal torus in $U(n)$ is the set

$$T = \left\{ \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \right\},$$

hence the rank of $U(n)$ is n. This is obviously a torus. To show that it is maximal, let $A \in U(n)$ be an element that commutes with T. Consider the subgroup T_j of T consisting of matrices with 1 in the j^{th} diagonal entry. Then, if $t_j \in T_j$, we have that $t_j A e_j = A t_j e_j = A e_j$ (here e_j is the column vector with 1 at the j^{th} place and 0 elsewhere). That is, Ae_j is left fixed by T_j, so $Ae_j = \lambda_j e_j$ for some complex number λ_j of modulus 1 (as $A \in U(n)$), therefore $\lambda_j = e^{i\phi_j}$. Since this is true for each j, this means that $A = \mathrm{diag}(e^{i\phi_1}, \dots, e^{i\phi_n})$, so $A \in T$ Hence T is maximal.

(2) A maximal torus of $SU(n)$ is the set

$$T = \left\{ \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \colon \theta_1 + \cdots + \theta_n = 0 \right\},$$

hence the rank of $SU(n)$ is $n - 1$. This is a torus because of the isomorphism $\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mapsto \mathrm{diag}(e^{i(\theta_1 - \theta_n)}, \dots, e^{i(\theta_{n-1} - \theta_n)})$ that maps T onto the maximal torus in $U(n-1)$. Maximality is shown as before.

Theorem 2.15 can be seen easily in this case: A standard result of linear algebra says that any $A \in SU(n)$ can be diagonalized, that is, there is a $U \in SU(n)$ with $UAU^{-1} \in T$.

(3) Let $\operatorname{rot}\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Then a maximal torus in $SO(2n+1)$ is the set of "block matrices" $\{\operatorname{diag}(\operatorname{rot}\theta_1, \dots, \operatorname{rot}\theta_n, 1)\}$. The rank of $SO(2n+1)$ is n.

(4) A maximal torus in $SO(2n)$ is the set $\{\operatorname{diag}(\operatorname{rot}\theta_1, \dots, \operatorname{rot}\theta_n)\}$, so its rank is n.

(5) A maximal torus in $Sp(n)$ is the set $\{\operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})\}$. Its rank is also n.

Notice that part (a) of Theorem 2.15 and the examples above justify the simplifications in the computations for the Killing form that we did in the previous section.

5. The classification of compact and connected Lie groups

All groups in this section are assumed to be compact and connected. We will present the classification theorem for such Lie groups. For a detailed presentation we refer to [**Brö-TD**].

Definition. A Lie group is called *simple* if it is non-abelian and it does not contain any proper normal Lie subgroups.

Equivalently, a Lie group is simple if its Lie algebra is simple, i.e. it is non-abelian and it has no proper ideals.

Theorem 2.17. (1) *Let G be a compact and connected Lie group. Then there exists a covering space of G that is isomorphic to the direct product of a torus and a compact, connected and simply connected Lie group.*

(2) *Every compact, connected and simply connected Lie group is isomorphic to the direct product of simple, compact, connected and simply connected Lie groups.*

(3) *The simple, compact, connected and simply connected Lie groups are the following:*

$$SU(n)\ (n \geq 2),\ \widetilde{SO}(2n+1)\ (n \geq 3),\ Sp(n)\ (n \geq 2),\ \widetilde{SO}(2n)\ (n \geq 4),$$

$$G_2,\quad F_4,\quad E_6,\quad E_7,\quad E_8.$$

Remarks.

(1) The group $\widetilde{SO}(n)$ is denoted by $Spin(n)$, and is the universal covering group of $SO(n)$. The spin groups are constructed by using the *Clifford algebras.* Spin groups are extremely interesting in particle physics. We refer to [**Fe**] and [**Si**] for an elementary presentation.

(2) The Lie algebras of the first four groups are denoted by A_{n-1}, B_n, C_n and D_n respectively. This is Cartan's classical notation. The following isomorphisms hold: $A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, $A_3 \cong D_3$, $D_2 \cong A_1 \oplus A_1$.

(3) The next five Lie groups in (3) of Theorem 2.17 are called the *exceptional* Lie groups. Their indices indicate the rank, and their dimensions are 14, 52, 78, 133, and 248 respectively. Each of these groups has an interesting reason for existing either by a special phenomenon in algebra or a special phenomenon in geometry. We refer to [**Jac**], [**Wan**] for more details on these.

(4) Concerning the idea of the classification, here is how it goes. We know that a simply connected Lie group is determined by its Lie algebra, so the compact semisimple Lie algebras are in one-to-one correspondence (up to isomorphism) with compact Lie groups. By complexifying these Lie algebras, we obtain a one-to-one correspondence between these, and the complex semisimple Lie algebras. However the complex semisimple Lie algebras are classified by their, still to come, root systems, and the root systems are classified by their bases. A simple description of the bases are the *Dynkin diagrams*, whose complete description is an elementary but nontrivial combinatorial problem. Hence we finally end-up with a one-to-one

correspondence between compact simply connected Lie groups and Dynkin diagrams.

In the next section we will give an overview of the structure of the complex semisimple Lie algebras.

6. Complex semisimple Lie algebras

The aim of this section is to present the central notions related to the structure of the complex semisimple Lie algebras. There are several references for more details, such as [**Hu**], [**Sa**], [**Se**], [**Si**], [**Va**], [**Wan**], to list a few.

Definitions. (a) Let $\mathfrak{g}$ be a complex Lie algebra. The *adjoint representation* of $\mathfrak{g}$ is the homomorphism $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(X)(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

(b) The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form given by $B(X,Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$ $(X, Y \in \mathfrak{g})$.

(c) The Lie algebra $\mathfrak{g}$ is called *semisimple* if its Killing form is non-degenerate.

(d) It is called *simple* if it is non-abelian and its only ideals are $\{0\}$ and $\mathfrak{g}$.

(e) A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$, such that for all $H \in \mathfrak{h}$ the endomorphism $\mathrm{ad}(H)$ is diagonalizable.

Remark. Part (c) in the above definition is actually an important theorem due to Cartan and Killing. The historic definition of a semisimple Lie algebra is a purely algebraic one, saying that its radical (its largest solvable ideal) is zero. We avoided this as we do not use these concepts in this book.

Proposition 2.18. *A Lie algbera is semisimple if and only if it is isomorphic to a product of simple algebras.*

Proposition 2.19. (a) *Any complex Lie algebra contains a Cartan subalgebra.*

(b) *Let G be the group of automorphisms of $\mathfrak{g}$ generated by the elements* $\exp(\mathrm{ad}X) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}X)^n$ $(X \in \mathfrak{g})$. *Then any* *two Cartan subalgebras are conjugate under G. This group is called the adjoint group.*

Part (b) looks a bit abstract. In the terminology we have developed in the previous sections, a simpler version of it says that if G is a compact Lie group with Lie algbera $\mathfrak{g}$, and $\mathfrak{h}_1, \mathfrak{h}_2$ are two Cartan subalgebras of $\mathfrak{g}$, then there exists a $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{h}_1 = \mathfrak{h}_2$. Hence we can give the following definition:

Definition. The *rank* of a Lie algebra is the dimension of a Cartan subalgebra.

From now on $\mathfrak{g}$ will be a complex semisimple Lie algebra and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Then for all $\alpha \in \mathfrak{h}^*$ denote by $\mathfrak{g}^\alpha$ the corresponding eigenspace of $\mathfrak{g}$, that is,

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \colon \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Any element $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq \{0\}$ is called a *root* of $\mathfrak{g}$. In this case, the set $\mathfrak{g}^\alpha$ is called the *root space* that corresponds to the root α. The set of all roots is denoted by R and is called the *root system* of $\mathfrak{g}$ (relative to $\mathfrak{h}$). In particular, $\mathfrak{g}^0$ is the set of all elements in $\mathfrak{g}$ that commute with $\mathfrak{h}$. Since $\mathfrak{h}$ is maximal Abelian, we know that $\mathfrak{g}^0 = \mathfrak{h}$. Furthermore, since the endomorphisms $\mathrm{ad}(H)$ are diagonalizable for all $H \in \mathfrak{h}$ and commute with each other, by a standard theorem of linear algebra they are simultaneously diagonalizable. Hence, we obtain the *root space decomposition* of $\mathfrak{g}$ (for a given $\mathfrak{h}$):

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^\alpha$$

For each $\alpha \in R$, let H_α denote the unique element in $\mathfrak{h}$ such that $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$. This is called the *root vector* for α. The root spaces have the following properties:

Proposition 2.20. (a) *If α is a root, then so is $-\alpha$.*

(b) *The roots span $\mathfrak{h}^*$ and the root vectors span $\mathfrak{h}$.*

(c) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ *If $\alpha + \beta \notin R$ the bracket is interpreted as* 0.

(d) *The Killing form is non-degenerate on $\mathfrak{h}$.*

(e) *The subspace $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ of $\mathfrak{h}$ has dimension* 1.

(f) *Let* $E_\alpha \in \mathfrak{g}^\alpha$ *and* $E_{-\alpha} \in \mathfrak{g}^{-\alpha}$ *Then* $[E_\alpha, E_{-\alpha}] = B(E_\alpha, E_{-\alpha})H_\alpha$.
(g) *For each* $\alpha \in R$ *the dimension of each* $\mathfrak{g}^\alpha$ *is* 1.
(h) *If* $\alpha \in R$ *and* $k\alpha \in R$ *for some integer* k, *then* $k = \pm 1$.

Elements E_α of $\mathfrak{g}^\alpha$ with $[E_\alpha, E_{-\alpha}] = H_\alpha$ (hence $B(E_\alpha, E_{-\alpha}) = 1$), are called *root elements.* Now let $\mathfrak{h}_\mathbb{R} = \sum_\alpha \mathbb{R}H_\alpha$ (the real subspace of $\mathfrak{h}$ formed by real linear combinations of the $H_\alpha, \alpha \in R$).

Proposition 2.21. (a) *The Killing form restricted to* $\mathfrak{h}_\mathbb{R}$, *is a real positive-definite bilinear form.*
(b) *Every root* α *takes real values when restricted to* $\mathfrak{h}_\mathbb{R}$.
(c) $\mathfrak{h}_\mathbb{R}$ *is a real form of* $\mathfrak{h}$, *that is* $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$.

We mention at this point that there are usually more roots than the dimension of $\mathfrak{g}$, i.e. the $\{H_\alpha\}$ are not linearly independent.

Since the Killing form B is non-degenerate on $\mathfrak{h}$, we have the usual isomorphism of $\mathfrak{h}$ with its dual $\mathfrak{h}^*$: for each $\lambda \in \mathfrak{h}^*$ there is a unique $H_\lambda \in \mathfrak{h}$ with $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{h}$. Then the real subspace $\mathfrak{h}_\mathbb{R}$ goes over $\mathfrak{h}^*_\mathbb{R}$ the $\mathbb{R}$-span of R. We then transfer the Killing form to $\mathfrak{h}^*$ (and to $\mathfrak{h}_\mathbb{R}$) by setting $(\lambda, \mu) = B(H_\lambda, H_\mu)$.

Proposition 2.22. (a) *The numbers* $N(\alpha,\beta) = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ *are integers whose only possible values are* $0, \pm 1, \pm 2, \pm 3$. *They are called the Cartan integers, and are usually put together to form the "Cartan matrix".*
(b) *For each* $\alpha \in R$ *we consider the reflection map* $S_\alpha \colon \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$ *with respect to the hyperplane orthogonal to* α, *given by* $S_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$. *Notice that* $S_\alpha(\alpha) = -\alpha$. *Then* $S_\alpha(R) = R$, *that is, the set of roots is invariant under all* S_α.

The set $\{S_\alpha \colon \alpha \in R\}$ generates a group of isometries of $\mathfrak{h}^*_\mathbb{R}$ called the *Weyl group* of R (or of $\mathfrak{g}$) with respect to $\mathfrak{h}$.

For any $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$, we have that $[E_\alpha, E_b] = N_{\alpha,\beta}E_{\alpha+\beta}$ for some complex number $N_{\alpha,\beta}$. These numbers determine the "multiplication table" of $\mathfrak{g}$ and are called the *structure constants* of $\mathfrak{g}$. They satisfy the following properties:

Proposition 2.23. (a) $N_{\alpha,\beta} = -N_{\beta,\alpha}$ $(\alpha, \beta \in R,\ \alpha + \beta \neq 0)$.

(b) $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$ $(\alpha, \beta, \gamma \in R,\ \alpha + \beta + \gamma = 0)$.

(c) $N_{\alpha,\beta}N_{\gamma,\delta} + N_{\alpha,\gamma}N_{\delta,\beta} + N_{\alpha,\delta}N_{\beta,\gamma} = 0$ $(\alpha, \beta, \gamma, \delta \in R,\ \alpha + \beta + \gamma + \delta = 0)$.

(d) *The $\alpha + \beta$ is a root if and only if $N_{\alpha,\beta}$ is not zero.*

(e) *It is possible to choose the root elements $\{E_\alpha\}$ in such a way, so that the structure constants are real numbers satisfying $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. However, something much stronger is true:*

(f) *(Chevalley) The structure constants can be chosen to be integers.*

The following definition summarizes the previous information.

Definition. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with $\mathfrak{h}$ a Cartan subalgebra, and root system R. Let $H_1, \ldots, H_l$ (l=rank of $\mathfrak{g}$) be a basis for $\mathfrak{h}$. For each $\alpha \in R$ let E_α be root elements (generators of $\mathfrak{g}^\alpha$) satisfying $[E_\alpha, E_{-\alpha}] = H_\alpha$, and such that the structure constants are integers with $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Then the set $\{H_1, \ldots, H_l; E_\alpha \colon \alpha \in R\}$ is said to be a *Weyl-Chevalley basis* for $\mathfrak{g}$.

Proposition 2.24. *Let R be the root system of a complex semisimple Lie algebra $\mathfrak{g}$ (with respect to a fixed Cartan subalgebra). Then there exists a subset $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ ($l =$ rank of $\mathfrak{g}$) such that every root $\alpha \in R$ can be expressed uniquely as $\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l$, where n_i are integers either all nonnegative or all nonpositive.*

Any such set Π is called a *set of simple roots* for R (the terms *fundamental system*, *simple system*, or *basis* are also used). A set of simple roots Π is called *irreducible* if there is no nontrivial disjoint union $\Pi = \Pi_1 \cup \Pi_2$ with $(\alpha, \beta) = 0$ for all $\alpha \in \Pi_1$ and $\beta \in \Pi_2$.

A root α is called *positive* ($\alpha > 0$) if $\alpha = \sum_i^l n_i\alpha_i$ with all $n_i \geq 0$. Let R^+ denote the set of all positive roots, and by $R^- = \{-\alpha \colon \alpha \in R^+\}$. The choice of the set R^+ is also called an *ordering* in R and satisfies the following properties:

(1) $R^+ \cap (-R^+) = \emptyset$, $R^+ \cup (-R^+) = R$,

(2) for each $a, \beta \in R^+$ with $\alpha + \beta \in R$, then $\alpha + \beta \in R^+$

This corresponds to the usual meaning, i.e., for each $\alpha, \beta \in R$, then $\alpha > \beta$ if and only if $\alpha - \beta \in R^+$

Now let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a set of simple roots for the set of roots R, and recall the number $N(\alpha, \beta) = \frac{2(\alpha,\beta)}{(\beta,\beta)}$.

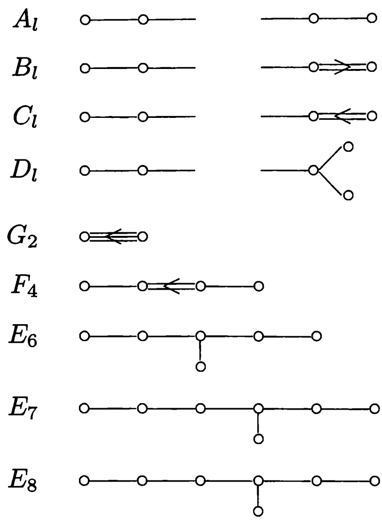
Definition. The *Dynkin diagram* of a root system R with a set of simple roots Π consists of a planar graph with l vertices labeled with $\alpha_1, \ldots, \alpha_l$, and $N(\alpha_i, \alpha_j)N(\alpha_j, \alpha_i)$ line segments joining the vertex α_i to the one α_j. If $N(\alpha, \beta) > 0$ and $(\beta, \beta) > (\alpha, \alpha)$, draw an arrow on the line segments from the vertex of β (long root) to the vertex of α (short root).

The fundamental result is the following:

Theorem 2.25. *Assigning to each complex semisimple Lie algebra the Dynkin diagram of the root system of a Cartan subalgebra, sets up a one-to-one correspondence between the set of such Lie algebras (up to isomorphism) and fundamental root systems (up to equivalence). In particular, the simple Lie algberas correspond to irreducible fundamental systems.*

Next, we list the simple complex Lie algebras and their corresponding Dynkin diagrams.

Name	Description	Rank	Dimension
A_l	$\mathfrak{sl}_{l+1}\mathbb{C}$	$l \geq 1$	$l(l+2)$
B_l	$\mathfrak{so}_{2l+1}\mathbb{C}$	$l \geq 2$	$l(2l+1)$
C_l	$\mathfrak{sp}_l\mathbb{C}$	$l \geq 3$	$l(2l+1)$
D_l	$\mathfrak{so}_{2l}\mathbb{C}$	$l \geq 4$	$l(2l-1)$
G_2	—	2	14
F_4	—	4	52
E_6	—	6	78
E_7	—	7	133
E_8	—	8	248



The Dynkin diagrams encode the combinatorial information of the root system. By some miracle, these same diagrams encode other objects, such as singularity types in algebraic geometry.

Definition. A real Lie algebra $\mathfrak{g}_0$ is called a *real form* of a complex Lie algbera $\mathfrak{g}$, if $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$, that is, $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$.

We remark that $\mathfrak{g}$ may have several non-isomorphic (over $\mathbb{R}$) real forms. For example, the real orthogonal Lie algebra $\mathfrak{o}(n) = \mathfrak{o}(n, \mathbb{R})$ is a real form of the complex orthogonal Lie algebra $\mathfrak{o}(n, \mathbb{C})$. However, the Lie algebra $\mathfrak{o}(p, q)$ consisting of the operators in $\mathbb{R}^n$ that leave the indefinite form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$ invariant, is also a real form of $\mathfrak{o}(n, \mathbb{C})$.

An important fact discovered by Weyl is that every complex semisimple Lie algebra has a compact real form. Compact means that its Killing form is negative definite. All compact real forms of $\mathfrak{g}$ are conjugate via an inner automorphism.

Any real form $\mathfrak{g}_{\mathbb{R}}$ can be characterized as the fixed point set of a conjugate-linear involution $\tau\colon \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of

$\mathfrak{g}$ considered as a real Lie algebra. (This is the complex conjugation with respect to the real form.) If $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^\alpha$ is a root space decomposition of $\mathfrak{g}$, then we may use it to construct a compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ as follows. The conjugate-linear map $\tau_0 \colon \mathfrak{g} \to \mathfrak{g}$ defined by

$$\tau_0|_{\mathfrak{h}_\mathbb{R}} = -\mathrm{Id}, \qquad \tau_0(E_\alpha) = -E_{-\alpha}$$

is called the *standard involution* associated with the root space decomposition. The set of fixed points of τ_0 is the desired compact real form. Explicitly,

$$\mathfrak{g}_0 = i\mathfrak{h}_\mathbb{R} \oplus \bigoplus_{\alpha \in R^+} \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \bigoplus_{\alpha \in R^+} \mathbb{R}(i(E_\alpha + E_{-\alpha})).$$

Another interesting fact is that the elements iH_α, $E_\alpha - E_{-\alpha}$ and $i(E_\alpha + E_{-\alpha})$ generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$. The isomorphism is given by

$$iH_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_\alpha - E_{-\alpha} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_\alpha + E_{-\alpha} \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Hence $\mathfrak{g}$ has many such subalgebras. We can also use the root spaces to obtain homomorphisms into $\mathfrak{sl}(2, \mathbb{C})$:

$$iH_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Example.

We now exhibit the notions discussed above for the case of $\mathfrak{g} = A_l = \mathfrak{sl}_{l+1}\mathbb{C}$. A Cartan subalgebra is the set

$$\mathfrak{h} = \{H = \mathrm{diag}(a_1 \cdots a_{l+1}) \colon a_i \in \mathbb{C}, \ \sum a_i = 0\}.$$

We write E_{ij} for the matrix whose (i,j)-entry is 1 and the other entries are zero. The matrices

$$E_{ii} - E_{i+1,i+1} \ (1 \le i \le l) \qquad E_{ij} \ (1 \le i \ne j \le l+1)$$

form a basis for $\mathfrak{g}$ with commutation rules $[E_{ij}, E_{jk}] = E_{ik}$ $(i,j,k$ distinct$)$. Let $\epsilon_1, \dots, \epsilon_{l+1}$ be the linear functionals on $\mathfrak{h}$ defined by $\epsilon_i(H) = a_i$. Since $[H, E_{ij}] = (a_i - a_j)E_{ij} = (\epsilon_i - \epsilon_j)(H)E_{ij}$, the set of roots is

$$R = \{\epsilon_i - \epsilon_j \colon i \ne j, \ 1 \le i,j \le l+1\}$$

and the E_{ij} $(i \ne j)$ are the root elements. The corresponding root spaces are $\mathfrak{g}^{\epsilon_i - \epsilon_j} = \mathbb{C}E_{ij}$. To compute the Killing form, we take two elements $H, H' \in \mathfrak{h}$ and by using the basis for $\mathfrak{g}$ described above, we obtain

$$\begin{aligned} B(H,H') = \sum_{i \ne j} (a_i - a_j)(a_i' - a_j') &= 2(l+1) \sum_{1 \le i \le l+1} a_i a_i' \\ &= 2(l+1)\,\mathrm{tr}(HH'). \end{aligned}$$

It follows that the root vectors are

$$H_{\epsilon_i - \epsilon_j} = \frac{1}{2(l+1)}(E_{ii} - E_{jj}).$$

Furthermore,

$$\begin{aligned} (\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_m) &= B(H_{\epsilon_i - \epsilon_j}, H_{\epsilon_k - \epsilon_m}) \\ &= \frac{1}{2(l+1)}\,\mathrm{tr}[(E_{ii} - E_{jj})(E_{kk} - E_{mm})], \end{aligned}$$

hence $(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j) = \frac{1}{l+1}$.

Now, let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \le i \le l)$. Then the set $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots, and the corresponding set of positive roots is $R^+ = \{\epsilon_i - \epsilon_j : i < j\}$. Hence, there are l vertices in the Dynkin diagram of $\mathfrak{g}$. The rank is of course l.

Finally, concerning the Weyl group, the reflection S_{12} corresponding to the root $\epsilon_1 - \epsilon_2$ consists of the interchange of the coordinates a_1 and a_2 of any $H \in \mathfrak{h}$. We conclude that the Weyl group of $\mathfrak{g}$ is the group S_{l+1}, the symmetric group of permutations of the set $\{1, 2, \ldots, l+1\}$.

Chapter 3

The Geometry of a Compact Lie Group

1. Riemannian manifolds: A review

In this chapter we will study the Riemannian geometry of a Lie group. That is to say, we will choose an appropriate Riemannian metric, and compute the various geometrical objects, such as curvature and geodesics. First we will give a summary of Riemannian manifolds. References for a first reading would include [**C-Ch-La**], [**DC**], [**Ga-Hu-La**], [**ON**], and [**Wil**].

Definition. A *Riemannian metric* on a smooth manifold M is a correspondence which associates to each point $p \in M$ an inner product $g_p = \langle\ ,\ \rangle_p$ (that is a symmetric bilinear, positive definite form) on the tangent space T_pM, which varies differentiably in the following sense: For every pair of smooth vector fields X, Y in a neighborhood of p, the map $p \mapsto \langle X_p, Y_p\rangle_p$ is smooth. A smooth manifold with a Riemannian metric is called a *Riemannian manifold*, and is denoted by (M, g).

Let $\frac{\partial}{\partial x_i}$ $(1 \leq i \leq n)$ be the coordinate vector fields at p in a local chart around p, and let $v, w \in T_pM$ with

$$v = \sum_{i=1}^{n} u^i \left.\frac{\partial}{\partial x_i}\right|_m \qquad w = \sum_{i=1}^{n} w^i \left.\frac{\partial}{\partial x_i}\right|_m$$

Then $g_p(v,w) = \sum_{i,j} g_{ij}(p)v^i w^j$, where

$$g_{ij}(p) = g\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p\right)$$

We use the notation $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$ with $g_{ij} = g_{ji}$ or simply $g = \sum_{i,j} g_{ij} dx^i dx^j$ If we extend the vectors v, w to corresponding vector fields V, W, then $g(V,W) = \langle V, W\rangle$ is a smooth real-valued function on M. In the language of tensors[1], g is a symmetric, non-degenerate $(0,2)$ tensor field on M.

We will now see when two Riemannian manifiolds are considered to be the same.

Definition. Let (M,g), (N,g') be Riemannian manifolds. An *isometry* is a diffeomorphism $f\colon M \to N$ that preserves the metrics, in the sense that

$$g_p(u,v) = g'_{f(p)}(df_p(u), df_p(v)) \qquad \text{for all } p \in M, u, v \in T_pM.$$

Two Riemannian manifolds are *isometric* if there is an isometry between them.

Loosely speaking, a geometrical object or quantity preserved (in an appropriate sense) by all isometries is called an *isometric invariant.* Riemannian geometry is traditionally described as the study of such invariants.

Examples.

(1) Let $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$. The metric is given by $g(e_i, e_j) = \delta_{ij}$. In this case $\mathbb{R}^n$ is called the *Euclidean space of dimension n.*

(2) Immersed manifolds. Let $f\colon M \to N$ be an immersion (that is smooth, with df_p one-to-one for all $p \in M$). If N has a Riemannian metric g', then f induces a Riemannian metric g on M by defining

[1] Tensors are the multilinear extension of vectors and their duals. Further details on the tensor algebra can be found in advanced linear algebra books. For details on tensor fields, see, for instance, Chapter 2 of [**ON**] or Chapter 1, Sec. 2 of [**He**].

$g_p(u, v) = g'_{f(p)}(df_p(u), df_p(v))$ $(u, v, \in T_pM)$. This metric on M is called the metric *induced* by f, and f is called an *isometric immersion.*

For example, the metric on the sphere $S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$ induced from the Euclidean metric from $\mathbb{R}^n$ is called the *canonical metric* or the *standard metric* on S^{n-1}. The induced metric on an immersed manifold is none other than the first fundamental form of classical differential geometry.

Let $\mathcal{X}(M)$ denote the set of all smooth vector fields of a manifold M

Definition. An (affine) *connection* ∇ on a smooth manifold M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

denoted by $(X, Y) \mapsto \nabla_X Y$ that satisfies the following conditions:

$$\begin{aligned} \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_{fX+gY} Z &= f\nabla_X Z + g\nabla_Y Z, \\ \nabla_X(fY) &= f\nabla_X Y + X(f)Y \qquad \text{(Leibniz rule)} \end{aligned}$$

for all $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{F}(M)$.

The connection is a way of taking covariant derivatives on a manifold. To be more precise, recall that if X is a vector field on $\mathbb{R}^n$ and V a vector at $p \in \mathbb{R}^n$, then the classical covariant (or directional) derivative of X at p in the direction of V is

$$\nabla_V X = \lim_{t \to o} \frac{X(p + tV) - X(p)}{t}.$$

The vector $X(p + tX)$ lies in $T_{p+tV}\mathbb{R}^n$, and the vector $X(p)$ lies in $T_p\mathbb{R}^n$. In order to be able to subtract these vectors, the tangent spaces that they lie in need to be identified with each other. This can be done in a canonical manner, as both tangent spaces are naturally isomorphic to $\mathbb{R}^n$. On a manifold the tangent space T_pM is always isomorphic to $\mathbb{R}^n$ but not canonically. So it is this concept of identifying tangent spaces at different points in order to take covariant derivatives[2], that is introduced with the above definition.

The following theorem is also referred to as the "miracle" of Riemannian geometry ([**ON**]).

Theorem 3.1. *Given a Riemannian manifold M, there exists a unique connection (called the Levi-Civita or Riemannian connection) such that*

$$(a)\ [X,Y] = \nabla_X Y - \nabla_Y X,$$
$$(b)\ X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X,Y,Z \in \mathcal{X}(M)$. This connection is characterized by the Koszul formula,

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ & - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle. \end{aligned}$$

Condition (b) has a special geometric significance which will be explained later on.

Definition. (a) A *curve* in a manifold M is a smooth map $\alpha\colon I \to M$, where I is an open interval in $\mathbb{R}$.

(b) A *vector field* along a curve $\alpha\colon I \to M$ is a smooth map that assigns to every $t \in I$ a tangent vector $V(t) \in T_{\alpha(t)}M$. To say that V is smooth means that for any smooth function f on M, the function $t \mapsto V(t)f$ is a smooth function on I.

For example, the velocity vector field $d\alpha(\frac{d}{dt})$, denoted by $\alpha'(t)$, is a vector field along α.

Given a vector field V along α, there is a natural way to define a vector rate of change $V'(t)$.

[2]The connection can be extended to a covariant differentiation of tensors (tensor fields) by using contractions and an extended Leibniz rule. For details, see the book by Do Carmo [**DC**] or the book by O'Neill [**ON**].

Proposition 3.2. *Let M be a Riemannian manifold with Riemannian connection ∇, and α a curve of M. Then there exists a unique operator that associates to a vector field V along the curve α another vector field $V'(t) = \frac{DV}{dt}$ along α, such that:*

(a) $\dfrac{D}{dt}(aV + bW) = a\dfrac{DV}{dt} + b\dfrac{DW}{dt} \qquad (a, b \in \mathbb{R}).$

(b) $\dfrac{D}{dt}(fV) = \dfrac{df}{dt}V + f\dfrac{DV}{dt} \qquad (f \in \mathcal{F}(I)).$

(c) *If V is induced by a vector field $Y \in \mathcal{X}(M)$, i.e., $V(t) = Y(\alpha(t))$, then* $\dfrac{DV}{dt} = \nabla_{\alpha'(t)}Y.$

(d) $\dfrac{d}{dt}\langle V, W\rangle = \langle\dfrac{DV}{dt}, W\rangle + \langle V, \dfrac{DW}{dt}\rangle.$

The vector field $V'(t)$ is called *the covariant derivative of V along α* or *induced covariant derivative.* In the special case that $\frac{DV}{dt} = 0$, the vector field V along α is called *parallel.* We can now give a geometric interpretation of condition (b) in Theorem 3.1. It can be shown that this is equivalent to the fact that for any smooth curve α and any pair of parallel vector fields V and W along α, we have $\langle V, W\rangle$=constant.

The following definition is motivated from the notion of a parallel vector field along a curve:

Definition. A *geodesic* in a Riemannian manifold M is a curve $\gamma\colon I \to M$ whose vector field γ' is parallel, that is,

$$\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = 0.$$

The following theorem gives the local existence and uniqueness of geodesics.

Theorem 3.3. *Let $p_0 \in M$. Then there exists an open set $p_0 \in U \subset M$, and $\varepsilon > 0$, such that, for $p \in M$ and $v \in T_pM$ with $|v| < \varepsilon$, there exists a unique geodesic $\gamma_v \quad (-1, 1) \to M$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$.*

A geodesic γ is *maximal* if the domain I_v of γ is as large as possible. That is, if $\tilde{\gamma} \quad J \to M$ is another geodesic with $\tilde{\gamma}(0) = p$

and $\tilde{\gamma}'(0) = v$, then $J \subset I_v$ and $\tilde{\gamma} = \gamma|_J$. In the following we will denote by γ_v the maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ $(v \in T_pM)$.

Let $v \in T_pM$ and suppose there exists a geodesic $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. By Theorem 3.3, such a geodesic is unique. Then the point $\gamma(1) \in M$ is denoted by $\exp_p(v)$ and the corresponding map $T_pM \to M$ is called the *exponential map*. The geodesic γ can be described by the formula

$$\gamma(t) = \exp_p(tv),$$

and hence the exponential map carries lines through the origin of T_pM to geodesics of M through p.

We will now discuss the notion of curvature. The Riemann curvature tensor is one of the basic invariants of a Riemannian manifold. Originally Riemann introduced the notion of the sectional curvature in a rather geometric manner, as an extension of the Gauss curvature for surfaces to arbitrary Riemannian manifolds. His definition was not a "workable" one. It took several years to reach a formulation that has the advantage of being easy to use to prove theorems, even though it is far from Riemann's original intuitive concept. Besides various references for Riemannian geometry such as [**DC**], [**Ga-Hu-La**], [**Ko-No**], [**ON**], [**Spi, Vol. II**], [**Wil**], we also refer to [**Kü**], and the article [**Ber1**] of M. Berger for interpretations of the various curvatures, on a Riemannian manifold.

Definition. Let M be a Riemannian manifold M with Levi-Civita connection ∇. The Riemann *curvature tensor* is the function

$$R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

Remarks. (1) The opposite sign convention for R is used quite often.

(2) In tensorial language R is a $(1,3)$-tensor field.

(3) Sometimes it is useful to introduce the $(0,4)$-tensor, also denoted by R, given by $R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$.

(4) If $x, y \in T_pM$, the linear operator $R_{xy}\colon T_pM \to T_pM$ that sends z to $R_{xy}z$ is called the *curvature operator.*

The following identities are the *symmetries* of the curvature:

Proposition 3.4. *Let $x, y, z, w \in T_pM$. Then*

(1) $R_{xy} = -R_{yx}$,

(2) $\langle R_{xy}v, w\rangle = -\langle R_{xy}w, v\rangle$,

(3) $\langle R_{xy}v, w\rangle = \langle R_{vw}x, y\rangle$,

(4) $R_{xy}z + R_{yz}x + R_{zx}y = 0$ *(first Bianchi identity).*

Properties (1) and (3) can be summarized by saying that the curvature tensor at a point $p \in M$ defines a symmetric bilinear form ρ on $\Lambda^2 T_pM$ given by $\rho(x \wedge y, z \wedge w) = R(x,y,z,w)$.

There is also another symmetry called the *second Bianchi identity* which requires more tensorial language to be stated. The covariant derivative ∇R of the curvature tensor can be thought of as a $(1,4)$-tensor that assigns to four vector fields Z, X, Y, V the vector field $(\nabla_Z R)_{XY}V = (\nabla_Z R)(X,Y)V$ Then

Proposition 3.5 (Second Bianchi identity). *If $x, y, z \in T_pM$, then*

$$(\nabla_z R)(x,y) + (\nabla_x R)(y,z) + (\nabla_y R)(z,x) = 0.$$

A simpler real-valued function that completely determines R is the sectional curvature.

Let Π be a two-dimensional subspace of T_pM and let $x, y \in \Pi$ be two linearly independent vectors. Then the number

$$K_p(x,y) = \frac{\langle R_{xy}x, y\rangle}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2}$$

does not depend on the choice of the vectors x, y. It is called the *sectional curvature* of Π at p.

The importance of the sectional curvature lies in the fact that it determines the curvature tensor, as shown in the following theorem.

Theorem 3.6. *The curvature tensor at a point p is uniquely determined by the sectional curvatures of all the two-dimensional subspaces Π of the tangent space T_pM at p.*

A Riemannian manifold is said to have *constant sectional curvature* (positive or negative) if K_p is a constant (positive or negative) for all planes Π in T_pM and for all points $p \in M$. The sphere S^n is such an example. If the sectional curvature is zero at every point, then the Riemannian manifold is said to be *flat*. The Euclidean space $\mathbb{R}^n$ is such an example.

The Riemann curvature tensor is a rather complicated object. Hence, we need to define simpler tensors that are related to it, such as the Ricci curvature and the scalar curvature. This can be done by various contractions of the curvature tensor which, of course, involves losing some information about the manifold. For more conceptual motivations for these curvatures we refer to [**Ber1**] and [**Ga-Hu-La**].

Definition. The *Ricci curvature* $\mathrm{Ric}(X,Y)$ of a Riemannian manifold M is the trace of the map $Z \mapsto R(X,Z)Y$

If $E_1,\ldots,E_n$ is an orthonormal basis of the tangent space T_pM at a point p, then the Ricci curvature is given by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} \langle R(X,E_i)Y, E_i\rangle.$$

This is a symmetric $(0,2)$-tensor. It can be viewed as a map $\mathrm{Ric}\colon TM \times TM \to \mathbb{R}$. Alternatively, the Ricci curvature can be defined as a map $r\colon TM \to TM$ by the formula

$$\mathrm{Ric}(X,Y) = \langle r(X), Y\rangle.$$

Since the sectional curvature determines the curvature tensor, it also determines the Ricci curvature. Indeed, by polarization[3] and

[3] The identity $R(X,Y) = \frac{1}{4}\{R(X+Y,X+Y) - R(X-Y,X-Y)\}$.

scalar multiplication, Ric can be reconstructed at each point p from its values $\mathrm{Ric}(X,X)$ on unit vectors at p. Now, if $E_1,\dots,E_n$ is an orthonormal basis of T_pM with $X=E_1$, then we obtain

$$\mathrm{Ric}(X,X)=\sum_{i=2}^{n}\langle R(X,E_i)X,E_i\rangle=\sum_{i=2}^{n}K(X,E_i).$$

If the Ricci curvature is identically zero, M is called *Ricci-flat.*

Definition. The *scalar curvature* S of M is the trace of the Ricci curvature. It is the function on M given by

$$\mathrm{S}(p)=\sum_{i\neq j}K(E_i,E_j)=2\sum_{i<j}K(E_i,E_j),$$

relative to an othonormal basis $\{E_1,\dots,E_n\}$ of T_pM.

The above expression does not depend on the choice of the basis on T_pM.

Remarks. (1) For a 2-dimensional manifold the curvature tensor is given by the scalar curvature.

(2) For a 3-dimensional manifold, the curvature tensor is given by the Ricci curvature. (We refer to [**Be**] and [**Ga-Hu-La**] for more comments on these remarks.)

(3) If (M,g) is a Riemannian manifold and $g'=cg$ (c a non-zero constant) is a *homothety* of the metric g, then it turns out that ([**ON**]) $\nabla'=\nabla$, $R'=cR$, $K'=c^{-1}K$, $\mathrm{Ric}'=\mathrm{Ric}$, and $\mathrm{S}'=c^{-1}\,\mathrm{S}$.

2. Left-invariant and bi-invariant metrics

Since a Lie group G is a smooth manifold as well as a group, it is customary to use Riemannian metrics that link the geometry of G with its group structure. These metrics have the property that the left translations $L_a\colon G\to G$ are isometries for all $a\in G$, and are called *left-invariant.* More precisely, we have:

Definition. A Riemannian metric on a Lie group G is called *left-invariant* if $\langle u,v\rangle_x=\langle(dL_a)_xu,(dL_a)_xv\rangle_{L_a(x)}$ for all $a,x\in G$ and

$u, v \in T_xG$. Similarly, a Riemannian metric is *right-invariant* if each $R_a\colon G \to G$ is an isometry.

Since the tangent space at any point can be translated to the tangent space at the identity element of the group, the above relation for left-invariance can be simply written as $\langle u, v\rangle = \langle dL_a(u), dL_a(v)\rangle$. Now, a left-invariant metric on G is essentially a scalar product on the Lie algebra $\mathfrak{g}$ of G. We have the following:

Proposition 3.7. *There is a one-to-one correspondence between left-invariant metrics on a Lie group G, and scalar products on its Lie algebra $\mathfrak{g}$ (or a scalar product on T_eG under the canonical isomorphism $\mathfrak{g} \ni X \mapsto X_e$).*

Proof. Let $\langle\ ,\ \rangle$ be a left-invariant metric on G, and let $X, Y \in \mathfrak{g}$. Then the function $\langle X, Y\rangle\colon G \to \mathbb{R}$ is constant on G. Indeed, because of the left-invariance of the vector fields X, Y as well as of the metric, we have that for any $a \in G$,

$$\begin{aligned}\langle X, Y\rangle(a) &= \langle X_a, Y_a\rangle = \langle dL_aX_e, dL_aY_e\rangle \\ &= \langle X_e, Y_e\rangle = \langle X, Y\rangle_e.\end{aligned}$$

Thus $\langle X, Y\rangle$ defines a scalar product on $\mathfrak{g}$. Conversely, if $\langle\ ,\ \rangle_e$ is a scalar product on $\mathfrak{g}$, then the metric defined by

$$\langle x, y\rangle_a = \langle (dL_{a^{-1}})_a x, (dL_{a^{-1}})_a y\rangle_e \qquad (a \in G,\ u, v \in T_aG),$$

is a left-invariant metric on G. $\square$

Definition. A metric on G that is both left-invariant and right-invariant is called *bi-invariant*

From Theorem 2.4, and the proof of Theorem 2.5 we obtain the following:

Theorem 3.8. *A compact Lie group possesses a bi-invariant metric.*

For the case of bi-invariant metrics, Proposition 3.7 extends as follows:

Proposition 3.9. *There is a one-to-one correspondence between bi-invariant metrics on G and* Ad*-invariant scalar products on $\mathfrak{g}$, that is $\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle$ for all $g \in G, X, Y \in \mathfrak{g}$. Furthermore, the last condition is equivalent to the relation*

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle.$$

Proof. We know that (cf. proof of Theorem 2.8) $\mathrm{Ad}(g)X = dR_{a^{-1}}X$ for all $a \in G, X \in \mathfrak{g}$ and hence, by using the right invariance,

$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle dR_{a^{-1}}X, dR_{a^{-1}}Y\rangle = \langle X, Y\rangle.$$

To show the next relation, let $\exp(tX)$ be the flow of X. Then

$$\begin{aligned}\langle [X,Y], Z\rangle &= \langle \mathrm{ad}_X Y, Z\rangle = \langle \frac{d}{dt}\mathrm{Ad}(\exp tX)Y\Big|_{t=0}, Z\rangle \\ &= \frac{d}{dt}\langle \mathrm{Ad}(\exp tX)Y, Z\rangle\Big|_{t=0} = \frac{d}{dt}\langle Y, \mathrm{Ad}(\exp(-tX))Z\rangle\Big|_{t=0} \\ &= \langle Y, -\mathrm{ad}_X Z\rangle = -\langle Y, [X,Z]\rangle,\end{aligned}$$

where we used the Ad-invariance of the inner product in the fourth equality. What we just proved is equivalent to $\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle$. □

Example.

By Proposition 2.10, the Killing form of a Lie group is Ad-invariant. Hence, by Theorem 2.13, if the Lie group G is compact and semisimple, the Killing form (actually, its negative) provides a bi-invariant Riemannian metric.

3. Geometrical aspects of a compact Lie group

Here we will examine various geometrical quantities on a Lie group G with a bi-invariant metric.

Proposition 3.10. *Let G be a Lie group with a bi-invariant metric. Then*

(a) *The Riemannian connection is given by $\nabla_X Y = \frac{1}{2}[X,Y]$ for all $X, Y \in \mathfrak{g}$.*

(b) *The geodesics of G starting at e are the one-parameter subgroups of G.*

Proof. (a) We saw in the proof of Proposition 3.7 that the function $\langle X, Y\rangle$ is constant, hence $Z\langle X, Y\rangle = 0$ for all $Z \in \mathfrak{g}$. This means that the first three terms in Koszul's formula (Theorem 3.1) vanish, so this reduces to

$$2\langle \nabla_X Y, Z\rangle = -\langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.$$

From Proposition 3.9 the first two summands cancel; hence $\langle \nabla_X Y, Z\rangle = \frac{1}{2}\langle Z, [X, Y]\rangle$, which gives the result.

(b) Let α be the one-parameter subgroup corresponding to the left-invariant vector field X. Then $\nabla_{\alpha'}\alpha' = \nabla_X X|_\alpha = 0$, thus α is a geodesic. □

We now come to curvature.

Proposition 3.11. *Let G be a Lie group with a bi-invariant metric. Then for any $X, Y, Z \in \mathfrak{g}$:*

(a) *The curvature tensor is given by*

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z].$$

(b) *The sectional curvature is given by*

$$K(X, Y) = \frac{1}{4}\frac{\langle [X, Y], [X, Y]\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}.$$

(c) *The Ricci curvature is given by*

$$\operatorname{Ric}(X, Y) = \frac{1}{4}\sum_i \langle [X, E_i], [Y, E_i]\rangle,$$

where $\{E_i\}$ is an orthonormal basis for $\mathfrak{g}$. Furthermore,

$$r(X) = -\frac{1}{4}\sum_i [[X, E_i], E_i].$$

(d) *If G is compact and the bi-invariant metric is the metric coming from the Killing form, then the scalar curvature is given by $\mathrm{S} = \frac{1}{4}\dim(G)$.*

Proof. (a) By Proposition 3.10 $\nabla_X Y = \frac{1}{2}[X,Y]$, hence by the definition of the curvature we obtain

$$R(X,Y)Z = \frac{1}{2}[[X,Y],Z] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]].$$

From the Jacobi identity $\mathcal{C}([X,[Y,Z]]) := [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (cyclic combination), so the last two summands give $-\frac{1}{4}[[X,Y],Z]$ from which the result is obtained.

(b) By the Ad-invariance of the inner product on $\mathfrak{g}$ (cf. Proposition 3.9), and part (a), we have

$$\langle R(X,Y)X,Y\rangle = \frac{1}{4}\langle [[X,Y],X],Y\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle.$$

(c) We compute

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \mathrm{tr}\{Z \mapsto R(X,Z)Y\} = \sum_i \langle R(X,E_i)Y,E_i\rangle \\ &= \frac{1}{4}\sum_i \langle [[X,E_i],Y],E_i\rangle = \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle,\end{aligned}$$

where we used the Ad-invariance in the last equality.

To show the expression for $r(X)$ we compute

$$\begin{aligned}\langle -\frac{1}{4}\sum_i [[X,E_i],E_i],Y\rangle &= -\frac{1}{4}\sum \langle [X,E_i],[E_i,Y]\rangle \\ &= \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle = \mathrm{Ric}(X,Y)\end{aligned}$$

for all $Y \in \mathfrak{g}$, from which the result is obtained.

(d) We compute

$$\begin{aligned}S = \mathrm{tr}\, r &= \sum_i \langle r(E_i),E_i\rangle = -\frac{1}{4}\sum_{i,j} \langle [[E_i,E_j],E_j],E_i\rangle \\ &= \frac{1}{4}\sum_{i,j} \langle [E_i,E_j],[E_i,E_j]\rangle = \frac{1}{4}\dim G.\end{aligned}$$

□

Definition. A Riemannian manifold (M, g) is called an *Einstein manifold* if the Ricci tensor satisfies the equation $\mathrm{Ric}(X, Y) = cg(X, Y)$ for some constant c.

Einstein metrics are privileged metrics on a Riemannian manifold for various reasons that we will discuss in Chapter 8. For the moment we have the following:

Proposition 3.12. *If G is semisimple and compact, and furnished with a bi-invariant metric, then*

$$\mathrm{Ric}(X, Y) = -\frac{1}{4}B(X, Y).$$

Thus, G is an Einstein manifold with respect to the Killing form metric.

Proof. By Proposition 3.11 and the definition of the Killing form we have

$$\begin{aligned} B(X, Y) &= \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y) = -\sum_i \langle [X, [Y, E_i]], E_i \rangle \\ &= -\sum_i \langle [Y, E_i], [X, E_i] \rangle = -4\,\mathrm{Ric}(X, Y). \quad \square \end{aligned}$$

We would like to stress that the above results are valid for a Riemannian metric on G which is bi-invariant, or equivalenly the corresponding inner product on $\mathfrak{g}$ is Ad-invariant. If the metric is simply left-invariant, then it is possible to obtain more general formulas (cf. [**Ch-Eb**]). For instance, the relation

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle$$

obtained in the proof of Proposition 3.10 gives

$$\nabla_X Y = \frac{1}{2}([X, Y] - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X),$$

where T^* denotes the adjoint of a linear operator T. Also, for geodesics with respect to a left-invariant metric we refer to [**Kaj**], [**Ma**], and [**Sz**].

Chapter 4

Homogeneous Spaces

1. Coset manifolds

Given a Lie group G and a closed subgroup K, it is possible to build a smooth manifold on the set $G/K = \{gK : g \in G\}$ of all left cosets of K in G. Furthermore, we will see that the group G acts in a natural way on G/K, and this action has the property that any two points in G/K can be joined by the action of G, i.e., the action is *transitive*. This manifold with this transitive action will be called a *homogeneous space*, and it includes a large variety of manifolds with special importance in mathematics and physics.

Consider the coset space G/K, and for later use denote the coset $eK = K$ by o. Let $\pi : G \to G/K$ denote the *projection* that sends each $g \in G$ to the coset gK. Also, for each $a \in G$ let $\tau_a : G/H \to G/H$ be the (left) *translation* that sends each gK to agK. If $a, b \in G$, and L_a is the left translation in G, we have

$$\pi \circ L_a = \tau_a \circ \pi, \qquad \tau_{ab} = \tau_a \circ \tau_b.$$

Proposition 4.1. *Let G be a Lie group, and K a closed subgroup of G. Then there is a unique way to make G/K a manifold so that the projection $\pi : G \to G/K$ is a submersion; that is, $d\pi_g$ is onto for all $g \in G$.*

For the proof of this proposition, as well as the other facts in this section, we refer to [**Br-Cl**], [**Ko-No**], [**War**]. The manifold G/K constructed in this way is called a *coset manifold.* Often in the literature G/K is called a homogeneous space, but sometimes this term is kept to mean a manifold M on which a Lie group G acts transitively, as we will see next. Indeed, we will see that this distinction is slight.

Definition. A left *action* of a group G on a manifold M is a smooth map $\lambda\colon G \times M \to M$ such that $\lambda(e, m) = m$ and $\lambda(ab, m) = \lambda(a, \lambda(b, m))$ for all $a, b \in G$ and $m \in M$.

We will denote $\lambda(a, m)$ by $a \cdot m$ or simply by am if there is no chance of confusion. Similarly, we can define a right action. A space M with an action of a group G is called a *G-space.* From now on, G will be a Lie group and M a smooth manifold.

If λ is an action of G on M, then for all $a \in G$ the map $\lambda_a\colon M \to M$ given by $\lambda_a(m) = \lambda(a, m)$ is a diffeomorphism of M, thus G is "represented" as a group of diffeomorphisms or "transformations" of M. For this reason the Lie group G is also referred to as a *transformation group* of the manifold M.

Definition. (a) An action is called *transitive* if for any $m, n \in M$ there exists a $g \in G$ such that $g \cdot m = n$.

(b) Let $m \in M$. The set $G_m = \{g \in G\colon g \cdot m = m\}$ is called the *isotropy group* or *isotropy subgroup* at m.

(c) The *orbit* of a point $m \in M$ is the set $G \cdot m = \{g \cdot m\colon g \in G\}$.

Let G/K be a coset manifold. Then the map $G \times G/K \to G/K$ that sends each (a, gK) to agK is called the *natural action* of G on G/K. This action is obviously transitive. We will see that every transitive action can be represented in this way.

Proposition 4.2. *Let $G \times M \to M$ be a transitive action of a Lie group G on a manifold M, and let $K = G_m$ be the isotropy subgroup of a point m. Then:*

(a) *The subgroup K is a closed subgroup of G.*

(b) *The natural map $j\colon G/K \to M$ given by $j(gK) = g\cdot m$ is a diffeomorphism. (In other words, the orbit $G\cdot m$ is diffeomorphic to G/K.)*

(c) *The dimension of G/K is* $\dim G - \dim K$.

Definition. A *homogeneous space* is a manifold M with a transitive action of a Lie group G. Equivalently, it is a manifold of the form G/K, where G is a Lie group and K a closed subgroup of G.

Now, let (M, g) be a Riemannian manifold. The set $I(M)$ of all isometries $M \to M$ forms a group under composition of functions. It is called the *isometry group* of M, and it is another geometric invariant of M. Roughly speaking, the larger $I(M)$ is, the simpler M is. We refer to [**Ga-Hu-La**], [**Ko-No**], [**ON**], and [**Oni**] for more discussion of the isometry group. Here we state the following important result:

Theorem 4.3 (Myers-Steenrod). *The isometry group of a Riemannian manifold is a Lie group.*

Definition. A *Riemannian homogeneous space* is a Riemannian manifold (M, g) on which its isometry group $I(M)$ acts transitively.

Proposition 4.4. *Let M be a Riemannian homogeneous manifold. Then the isotropy subgroup of a given point is a compact subgroup of $I(M)$. Furthermore, $I(M)$ is compact if and only if M is compact.*

Hence, a Riemannian homogeneous space M is diffeomorphic to a homogeneous space G/K, where $G = I(M)$ and K is the isotropy subgroup of a point.

Remarks.

(1) The presentation of a Riemannian homogeneous space M in the form G/K follows Klein's Erlangen program in spirit, in which the various non-Euclidean geometries were recognized as various examples of coset spaces G/K of Lie groups. More precisely, according to Klein, a geometry is a connected manifold M with a Lie group G acting transitively on it. Then all the properties of figures studied in the geometry remain invariant under G. The diffeomorphism given

in part (b) of Proposition 4.2 says, in Klein's terms, that instead of describing a geometry with base point m as a pair (M, m) together with the Lie group G, we can describe it equivalently as the pair (G, K), where K is the isotropy subgroup of m. Hence, several problems about G/K are formulated in terms of G and K, and then in terms of their corresponding infinitesimal objects $\mathfrak{g}$ and $\mathfrak{k}$. As a result, in many instances difficult non-linear problems (e.g. from differential equations) reduce to algebraic problems.

(2) There may be more than one Lie group acting transitively on a given Riemannian homogeneous space, so that a manifold may appear as a Riemannian homoneneous space under different groups (subgroups of $I(M)$). This will be evident from the next examples. However, we refer to [**Be, pp. 178-180**] and [**Ga-Hu-La, p. 63**] for a deeper discussion of this.

Examples.

(1) A Lie group is a homogeneous space in several ways. Here are two: $G = G \times G/G = G/\{e\}$. For the first representation of G as a homogeneous space, $G \times G$ acts on G by left and right translations, and the isotropy subgroup is G diagonally embedded in $G \times G$.

(2) *Spheres.* The group $SO(n+1)$ acts on the unit sphere S^n in $\mathbb{R}^{n+1}$ by restriction of the natural action of $\mathrm{GL}_{n+1}\mathbb{R}$ on $\mathbb{R}^{n+1}$. This action is transitive: if $x, y \in S^n$, and if $\{x, a_1, a_2, \ldots, a_n\}$ and $\{y, b_1, b_2, \ldots, b_n\}$ are two orthonormal bases of $\mathbb{R}^{n+1}$ inducing the same orientation, then the transition matrix lies in $SO(n+1)$. The isotropy subgroup of $(1, 0, \ldots, 0) \in S^n$ consists of all elements in $SO(n+1)$ of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in SO(n)$. This subgroup is identified with $SO(n)$, hence S^n is diffeomorphic to $SO(n+1)/SO(n)$, which we write as $S^n = SO(n+1)/SO(n)$. If we neglect the orientation of bases in $\mathbb{R}^{n+1}$, we obtain the alternative expression $S^n = O(n+1)/O(n)$.

(3) The complex and quaternionic analogues of Example 2 are $S^{2n+1} = SU(n+1)/SU(n) = U(n+1)/U(n)$ and $S^{4n+3} = Sp(n+1)/Sp(n)$. Notice that $S^0 = O(1)$, $S^1 = U(1)$, and $S^3 = Sp(1)$.

(4) *Projective spaces.* The real projective space $\mathbb{R}P^n$ is diffeomorphic to $SO(n+1)/O(n) = O(n+1)/O(n) \times O(1)$, and the complex projective space $\mathbb{C}P^n$ is diffeomorphic to $SU(n+1)/U(n) = U(n+1)/U(n) \times U(1)$.

(5) *Grassmann manifolds.* Let $Gr_k\mathbb{R}^n$ denote the set of all k-dimensional subspaces in $\mathbb{R}^n$ (such a subspace is called a *k-plane*). The group $O(n)$ acts naturally on $Gr_k\mathbb{R}^n$ by matrix multiplication. This action is transitive: Let V be the subspace of $\mathbb{R}^n$ spanned by the first k vectors of the canonical basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ Let $W \in Gr_k\mathbb{R}^n$, and choose an orthonormal basis $e'_1, \ldots, e'_n$ of $\mathbb{R}^n$ whose first k vectors span W Then, if A is the matrix that corresponds to the linear map that sends each e_i to e'_i, then $A \in O(n)$ and $AV = W$ The isotropy subgroup of the subspace V consists of the set of matrices

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

with $B \in O(k)$ and $C \in O(n-k)$, thus $Gr_k\mathbb{R}^n = O(n)/O(k) \times O(n-k)$. Furthermore, $SO(n)$ also acts transitively on $Gr_k\mathbb{R}^n$, hence $Gr_k\mathbb{R}^n = SO(n)/S(O(k) \times O(n-k))$. In particular, $\mathbb{R}P^n = SO(n+1)/S(O(n) \times O(1))$. Here, $S(O(k) \times O(l))$ denotes the subgroup of $\mathrm{SO}(k+l)$ consisting of matrices of the form $h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. and $\det(h) = 1$.

(6) The complex analogue of Example 5 is $Gr_k\mathbb{C}^n = SU(n)/S(U(k) \times U(n-k))$ with $\mathbb{C}P^n = SU(n+1)/S(U(n) \times U(1))$ as a special case.

(7) *Stiefel manifolds.* A *k-frame* in $\mathbb{R}^n$ is a set of k linearly independent, orthonormal vectors in $\mathbb{R}^n$. Let $V_k\mathbb{R}^n$ denote the set of all k-frames in $\mathbb{R}^n$. It can be shown (**[Br-Cl, p. 92, p. 252]**) that $V_k\mathbb{R}^n$ admits a smooth structure and is diffeomorphic to $SO(n)/SO(n-k) = O(n)/O(n-k)$. Some special cases are $V_1\mathbb{R}^n = S^{n-1}$, $V_n\mathbb{R}^n = SO(n)$, and $V_2\mathbb{R}^n = T_1S^{n-1}$, the unit tangent bundle.

(8) The complex analogue of Example 7 are the complex Stiefel manifolds $V_k\mathbb{C}^n = SU(n)/SU(n-k) = U(n)/U(n-k)$.

(9) *Symmetric spaces.* These are homogeneous spaces of special importance, which will be examined in Chapter 6. Examples 4–6 above are symmetric spaces.

(10) *Flag manifolds.* A *(full) flag* in $\mathbb{C}^n$ is an increasing collection $\mathbf{x} = \{V_1 \subset V_2 \subset \quad \subset V_{n-1}\}$ of complex subspaces V_i of $\mathbb{C}^n$ with $\dim V_i = i$. Let F_n be the set of all flags in $\mathbb{C}^n$. The group $SU(n)$ acts on F_n by $g \quad \mathbf{x} = \{gV_1 \subset gV_2 \subset \quad \subset gV_{n-1}\}$. This action is transitive: Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and let $\mathbf{x}^0$ be the flag obtained by setting $V_i^0 = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_i\}$. If $\mathbf{x} = \{V_1 \subset V_2 \subset \quad \subset V_{n-1}\}$ is an arbitrary flag, then let v_1 be a unit vector in V_1. If $v_1, \ldots, v_k$ have been defined, let v_{k+1} be a unit vector in V_{k+1} orthogonal to V_k. In this way we have obtained a set $v_1, \ldots, v_{n-1}$ of orthonormal unit vectors. Let v_n be the unit vector orthogonal to V_{n-1} and so that if $v_i = \sum a_{ij} e_j$, then $g = (a_{ij})$ is in $SU(n)$. Then $g \cdot \mathbf{x}^0 = \mathbf{x}$, so the action is transitive. Furthermore, the isotropy subgroup of $\mathbf{x}^0$ consists of all diagonal matrices in $SU(n)$, which is a maximal torus in $SU(n)$. Thus $F_n = SU(n)/S(U(1) \times \quad \times U(1))$ (n times), and this manifold is called a (full) flag manifold. More generally, a flag manifold is a homogeneous space of the form G/T, where G is a semisimple, compact Lie group and T a maximal torus in G.

(11) *Generalized flag manifolds.* The previous example can be generalized as follows. Let $n_1, \ldots, n_s$ be a set of positive integers with $n_1 + \cdots + n_s = n$, and let $F(n_1, \ldots, n_s)$ be the set of all *partial flags* $\mathbf{x} = \{V_1 \subset \quad \subset V_s\}$ with $\dim V_i = n_1 + \cdots + n_i$. The set $SU(n)$ acts on $F(n_1, \ldots, n_s)$ as in the previous example, the action is transitive and the isotropy subgroup of a fixed point is $S(U(n_1) \times \quad \times U(n_s))$, the group of matrices of the form $\mathrm{diag}(A_1, \ldots, A_s)$ with $A_i \in U(n_i)$ and $\det(A_1) \cdots \det(A_s) = 1$. Thus $F(n_1, \ldots, n_s) = SU(n)/S(U(n_1) \times \quad \times U(n_s))$, where the set $S(U(n_1) \times \quad \times U(n_s))$ is the centralizer of a torus in $SU(n)$. This manifold is called a *complex flag manifold.* More generally, a generalized flag manifold is a homogeneous space of the form $G/C(T)$, where G is a compact and semisimple Lie group, and $C(T)$ is the centralizer of a torus (not necessarily maximal) in G. The projective space $\mathbb{C}P^n$ and the Grassmann manifolds $Gr_k\mathbb{C}^n$ are special cases of generalized flag manifolds.

Generalized flag manifolds will be examined in more detail in Chapter 7.

2. Reductive homogeneous spaces

Let G/K be a homogeneous space and recall the projection $\pi\colon G \to G/K$, $\pi(g) = gK$. We will compute the differential $d\pi_e\colon \mathfrak{g} \to T_o(G/K)$, where $o = \pi(e) = K$. Let $X \in \mathfrak{g}$ and $\exp tX$ be the corresponding one-parameter subgroup. Then

$$d\pi_e(X) = \frac{d}{dt}(\pi \circ \exp tX)\Big|_{t=0} = \frac{d}{dt}((\exp tX)K)\Big|_{t=0}$$

From this we obtain that $d\pi_e(\mathfrak{k}) = 0$, that is, $\ker d\pi_e = \mathfrak{k}$, hence since $d\pi$ is onto (cf. Proposition 1.4), we get the canonical isomorphism

$$\mathfrak{g}/\mathfrak{k} \cong T_o(G/K).$$

In general, for any $X \in \mathfrak{g}$ we can define a vector field X^* on G/K by the formula

$$X^*_{gK} = \frac{d}{dt}(\exp tX)gK\Big|_{t=0}$$

Notice the formula $[X^*, Y^*] = -[X, Y]^*$

Now, we will consider the following important special case. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively.

Definition. A homogeneous space is called *reductive* if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$, that is, $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant.

The condition $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ implies that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. The converse is true if K is connected. Notice that $\mathfrak{m}$ need not be closed under bracket, as $\mathfrak{k}$ is. Hence, as an immediate consequence of the above isomorphism, if G/K is reductive, we have the canonical isomorphism

$$\mathfrak{m} \cong T_o(G/K).$$

For example, if G is a compact Lie group, then G/K is reductive, because we can take $\mathfrak{m} = \mathfrak{k}^\perp$ with respect to an Ad-invariant inner product on $\mathfrak{g}$. Actually, it can be shown that the above definition is not very restrictive: any homogeneous space that admits a G-invariant metric (see next chapter) is reductive. We refer to [**Kow-Sz**] for a detailed proof of this.

Examples.

(1) Let $G/K = SU(3)/S(U(1) \times U(1) \times U(1))$ (a flag manidold). The Killing form of $\mathfrak{su}(3)$ is $B(X,Y) = 6\,\mathrm{tr}\,XY$, and $\mathfrak{k}$ is the set $\{\mathrm{diag}(ia, ib, ic) \colon a+b+c = 0\}$. Then, with respect to B, the subspace $\mathfrak{m} = \mathfrak{k}^{\perp}$ is the set

$$\left\{ \begin{pmatrix} 0 & a_1 + ib_1 & a_2 + ib_2 \\ -a_1 + ib_1 & 0 & a_3 + ib_3 \\ -a_2 + ib_2 & -a_3 + ib_3 & 0 \end{pmatrix} \quad a_i, b_i \in \mathbb{R} \right\}$$

(2) Let $G/K = V_2\mathbb{R}^4 = SO(4)/SO(2)$ (a Stiefel manifold). The Killing form of $\mathfrak{so}(4)$ is $B(X,Y) = 2\,\mathrm{tr}\,XY$ Then

$$\mathfrak{k} = \mathfrak{o}(2) \cong \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{o}(2) \end{pmatrix}$$

and, with respect to B,

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & 0 \\ -a_{14} & -a_{24} & 0 & 0 \end{pmatrix} \quad a_{ij} \in \mathbb{R} \right\}$$

3. The isotropy representation

The adjoint representation $\mathrm{Ad} \equiv \mathrm{Ad}^G$ of a Lie group is related to the isotropy representation of a homogeneous space G/K. Recall the diffeomorphism $\tau_a \colon G/K \to G/K$ given by $\tau_a(gK) = agK$ $(a \in G)$. Let o be the base-point, corresponding to the coset eK. Then, for $k \in K$, $\tau_k(o) = o$.

Definition. The *isotropy representation* of the homogeneous space G/K (or simply of K) is the homomorphism

$$\mathrm{Ad}^{G/K} \quad K \to \mathrm{Gl}(T_oG/K)$$

defined by $k \mapsto (d\tau_k)_o$. More explicitly, it is given by

$$\mathrm{Ad}^{G/K}(k)(X) = (d\tau_k)_o(X) \qquad \text{for all } X \in T_o(G/K).$$

Next, we will explain the precise relation between Ad^G and $\mathrm{Ad}^{G/K}$ when the homogeneous G/K space is reductive, with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. First, let us summarize the various representations that we have defined so far:

$$\begin{aligned} \mathrm{Ad}^G &: G \to \mathrm{Aut}(\mathfrak{g}) \\ \mathrm{Ad}^K &\ \ K \to \mathrm{Aut}(\mathfrak{k}) \\ \mathrm{Ad}^{G/K} &: K \to \mathrm{Aut}(\mathfrak{m}). \end{aligned}$$

If we restrict Ad^G to K, we obtain the representation $\mathrm{Ad}^G\Big|_K \ K \to \mathrm{Aut}(\mathfrak{g})$. Since K is a Lie subgroup of G, then, for $k \in K$, $\mathrm{Ad}^K(k) = \mathrm{Ad}^G(k)\Big|_{\mathfrak{k}}$.

Proposition 4.5. *Let G/K be a reductive homogeneous space. Let $k \in K, X \in \mathfrak{k}$ and $Y \in \mathfrak{m}$. Then*

$$\mathrm{Ad}^G(k)(X+Y) = \mathrm{Ad}^K(k)X + \mathrm{Ad}^{G/K}(k)Y;$$

that is, the restriction $\mathrm{Ad}^G\Big|_K$ splits into the sum $\mathrm{Ad}^K \oplus \mathrm{Ad}^{G/K}$

Proof. Since the sum is direct, it suffices to prove the above equality for (a) $X = 0$ and (b) $Y = 0$. Case (b) is obvious, since this says that $\mathrm{Ad}^G(k)X = \mathrm{Ad}^K(k)X$. In case (a) we need to show that $\mathrm{Ad}^G(k)Y = \mathrm{Ad}^{G/K}(k)Y$ which is equivalent to $\mathrm{Ad}^G(k)Y = (d\tau_k)_o(Y)$. In other words, we need to show that the isotropy representation of G/K is equivalent to the adjoint representation of K in $\mathfrak{m}$, so by the definition of the equivalence of two representations (cf. Chapter 2, Section 1) it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\mathrm{Ad}^G(k)} & \mathfrak{m} \\ {\scriptstyle d\pi_e|_{\mathfrak{m}}}\downarrow & & \downarrow{\scriptstyle d\pi_e|_{\mathfrak{m}}} \\ T_o(G/K) & \xrightarrow{(d\tau_k)_o} & T_oG/K \end{array}$$

The upper horizontal map is obtained from the restriction of $\mathrm{Ad}^G(k)$ on $\mathfrak{m}$ and the reductivity property $\mathrm{Ad}^G(k)\mathfrak{m} \subset \mathfrak{m}$. Each of the vertical maps is the canonical isomorphism of $\mathfrak{m}$ with T_oG/K obtained from the restriction of $d\pi_e\colon \mathfrak{g} \to T_o(G/K)$ in $\mathfrak{m}$. So, let $k \in K$ and $Y \in \mathfrak{m}$ and calculate

$$\begin{aligned}(d\pi_e)_o(\mathrm{Ad}(k)Y) &= \frac{d}{dt}\exp(t\mathrm{Ad}(k)Y)K\Big|_{t=0} = \frac{d}{dt}\exp(\mathrm{Ad}(k)tY)K\Big|_{t=0} \\ &= \frac{d}{dt}(k\exp(tY)k^{-1})K\Big|_{t=0} = (d\tau_k)_o\,\frac{d}{dt}(\exp tY)K\Big|_{t=0} \\ &= (d\tau_k)_o(d\pi_e)(Y),\end{aligned}$$

where the third equality was obtained from Proposition 1.13 applied to the automorphism $g \mapsto xgx^{-1}$ of G. $\square$

The following corollary is immediate from the above proof.

Corollary 4.6. *The isotropy representation of G/K is equivalent to the adjoint representation of K in $\mathfrak{m}$.*

Remark. If the homogeneous space is not reductive, the above corollary is still true with $\mathfrak{m}$ replaced by $\mathfrak{g}/\mathfrak{k}$ (see [**Ga-Hu-La, Proposition 2.41**]).

Definition. A homogeneous space is called *isotropy irreducible* if the isotropy representation is irreducible (as a real representation).

Examples.

We refer to the formulas at the end of Section 2 of Chapter 2.

(1) We will compute the isotropy representation of the sphere $S^n = SO(n+1)/SO(n)$. Recall the standard representation $\lambda_n\colon SO(n) \to SO(n)$ of $SO(n)$, and the fact that $\mathrm{Ad}^{SO(n)} = \wedge^2\lambda_n$ (cf. Chapter 2, Section 2). We compute

$$\begin{aligned}\mathrm{Ad}^{SO(n+1)}\Big|_{SO(n)} &= \wedge^2\lambda_{n+1}\big|_{SO(n)} = \wedge^2(\lambda_n \oplus 1) \\ &= \wedge^2\lambda_n \oplus \wedge^2 1 \oplus (\lambda_n \otimes 1).\end{aligned}$$

The first summand corresponds to the adjoint representation of $SO(n)$, and the second summand is zero (the trivial representation 1 is one-dimensional). Hence, by Proposition 4.5, the isotropy representation of the sphere corresponds to the third summand, which is identified with λ_n, that is, $\mathrm{Ad}^{S^n} = \lambda_n$. This is irreducible.

(2) Let $G/K = O(n+1)/O(1)\times O(n)$ be the real projective space $\mathbb{R}P^n$. Then

$$\begin{aligned}\mathrm{Ad}^{O(n+1)}\Big|_{O(1)\times O(n)} &= \wedge^2\lambda_{n+1}\big|_{O(1)\times O(n)} = \wedge^2(\lambda_1 \oplus \lambda_n)\\ &= \wedge^2\lambda_1 \oplus \wedge^2\lambda_n \oplus (\lambda_1 \otimes \lambda_n).\end{aligned}$$

The first summand is zero, the second corresponds to $\mathrm{Ad}^{O(n)}$, thus $\mathrm{Ad}^{G/K} = \lambda_1 \otimes \lambda_n$. This is irreducible.

(3) Let $G/K = SO(2n)/U(n)$ (a symmetric space). Here it is more convenient to compute the complexified isotropy representation. We compute

$$\begin{aligned}\mathrm{Ad}^{SO(2n)} \otimes \mathbb{C}\Big|_{U(n)} &= \wedge^2(\lambda_{2n}\otimes\mathbb{C})\big|_{U(n)} = \wedge^2(\lambda_{2n}\otimes\mathbb{C}|_{U(n)})\\ &= \wedge^2(\mu_n \oplus \bar{\mu}_n) = \wedge^2\mu_n \oplus \wedge^2\bar{\mu}_n \oplus (\mu_n \otimes \bar{\mu}_n).\end{aligned}$$

The last summand is the complexified adjoint representation of $U(n)$, thus $\mathrm{Ad}^{SO(2n)/U(n)} \otimes \mathbb{C} = \wedge^2\mu_n \oplus \wedge^2\bar{\mu}_n$. This is irreducible.

(4) Let $G/K = U(3)/U(1)\times U(1)\times U(1)$ (a flag manifold of dimension 6). Let $\sigma_i\colon U(1)\times U(1)\times U(1) \to U(1)$ be the projection onto the i factor. Then

$$\begin{aligned}&\mathrm{Ad}^{U(3)} \otimes \mathbb{C}\Big|_K = \bar{\mu}_3 \otimes \mu_3|_K = \bar{\mu}_3|_K \otimes \mu_3|_K\\ &= \overline{\sigma_1 \oplus \sigma_2 \oplus \sigma_3} \otimes (\sigma_1 \oplus \sigma_2 \oplus \sigma_3) = (\bar{\sigma}_1 \oplus \bar{\sigma}_2 \oplus \bar{\sigma}_3) \otimes (\sigma_1 \oplus \sigma_2 \oplus \sigma_3)\\ &= (\bar{\sigma}_1\sigma_1 \oplus \bar{\sigma}_2\sigma_2 \oplus \bar{\sigma}_3\sigma_3) \oplus \Big(\bigoplus_{1\le i\ne j\le 3} \bar{\sigma}_i \otimes \sigma_j\Big),\end{aligned}$$

where $\bar{\sigma}_i\sigma_j$ means $\bar{\sigma}_i \otimes \sigma_j$. Therefore,

$$\mathrm{Ad}^{G/K} = \bar{\sigma}_1\sigma_2 \oplus \bar{\sigma}_1\sigma_3 \oplus \bar{\sigma}_2\sigma_3 \oplus \bar{\sigma}_2\sigma_1 \oplus \bar{\sigma}_3\sigma_1 \oplus \bar{\sigma}_3\sigma_2.$$

This is not irreducible. In fact, it is the direct sum of six one-dimensional inequivalent complex representations. Hence, the complexified tangent space of G/K is isomorphic to the direct sum of six one-dimensional complex subspaces

$$\mathfrak{m}^{\mathbb{C}} = K_{12} \oplus K_{13} \oplus K_{23} \oplus K_{21} \oplus K_{31} \oplus K_{32}.$$

This splitting determines a splitting of $\mathfrak{m}$ into a direct sum of three (irreducible) real subspaces each of dimension two, that is,

$$\mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23},$$

where $\mathfrak{m}_{ij}^{\mathbb{C}} = K_{ij} \oplus K_{ji}$.

(5) Let $G/K = SO(4)/SO(2)$ (a Stiefel manifold of dimension 5). We compute

$$\begin{aligned}\mathrm{Ad}^{SO(4)}\Big|_{SO(2)} &= \wedge^2\lambda_n\big|_{SO(2)} = \wedge^2(\lambda_2 \oplus 2)\\ &= \wedge^2\lambda_2 \oplus \wedge^2 2 \oplus (\lambda_2 \otimes 2)\\ &= \wedge^2\lambda_2 \oplus 1 \oplus \lambda_2 \oplus \lambda_2.\end{aligned}$$

The first summand is the adjoint representation of $SO(2)$, thus

$$\mathrm{Ad}^{SO(4)/SO(2)} = 1 \oplus \lambda_2 \oplus \lambda_2.$$

This is a direct sum of three irreducible representations of dimensions 1, 2 and 2 respectively. The last two are equivalent representations. We also remark that this decomposition is not unique, and the normalizer of K in G rotates one decomposition into another. This direct sum induces the direct sum

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

of $\mathfrak{m}$ into three irreducible subspaces of (real) dimensions 1, 2 and 2 respectively.

Chapter 5

The Geometry of a Reductive Homogeneous Space

1. G-invariant metrics

Let $(M = G/K, g)$ be a Riemannian homogeneous space. We assume that M is reductive with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is identified with the tangent space T_oM $(o = eK)$. According to Corollary 4.6 the isotropy representation of G/K is equivalent to the adjoint representation of K in $\mathfrak{m}$. As a consequence of this, many geometrical questions about M may be reformulated in terms of the pair (G, K) of Lie groups, and then in terms of the corresponding pair $(\mathfrak{g}, \mathfrak{k})$ of Lie algebras.

Since $M = G/K$ is a Riemannian homogeneous space, it admits a metric g which is G-invariant. More precisely, we have:

Definition. Let $M = G/K$ be a homogeneous space. A metric g on M is called *G-invariant* if for each $a \in G$ the diffeomorphism τ_a that sends $p \in M$ to $a \cdot p$ is an isometry; that is, $g(X, Y) = g(d\tau_a(X), d\tau_a(Y))$ for all $X, Y \in T_o(G/K)$ $(a \in G)$.

The next proposition gives a simple description of G-invariant metrics on a homogeneous space.

Proposition 5.1. *Let G/K be a homogeneous space. Then there is a one-to-one correspondence between:*

(1) *G-invariant Riemannian metrics g on G/K;*

(2) $\mathrm{Ad}^{G/K}$*-invariant scalar products $\langle\ ,\ \rangle$ on $\mathfrak{m}$; that is, $\langle X, Y\rangle = \langle \mathrm{Ad}^{G/K}(k)X, \mathrm{Ad}^{G/K}(k)Y\rangle$ for all $X, Y \in \mathfrak{m}$, $k \in K$; and*

(3) *(if K is compact and $\mathfrak{m} = \mathfrak{k}^{\perp}$ with respect to the negative of the Killing form B of G) $\mathrm{Ad}^{G/K}$-equivariant and B-symmetric operators $A\colon \mathfrak{m} \to \mathfrak{m}$ such that $\langle X, Y\rangle = B(AX, Y)$.*

We say that the scalar product is $\mathrm{Ad}^{G}(K)$*-invariant or simply* $\mathrm{Ad}(K)$*-invariant.*

Proof. Given a G-invariant metric g on G/K, by restricting to the tangent space at o we get a scalar product on $\mathfrak{m}$. Due to the commutativity of the diagram in the proof of Proposition 4.5, this product is $\mathrm{Ad}^{G/K}$-invariant. Conversely, let $\langle\ ,\ \rangle_o$ be an $\mathrm{Ad}^{G/K}$-invariant scalar product on $\mathfrak{m} \cong T_o(G/K)$. We extend this product at any point $aK \in G/K$ by setting

$$\langle X, Y\rangle_{aK} = \langle d\tau_{a^{-1}}(X), d\tau_{a^{-1}}(Y)\rangle_o.$$

This definition does not depend on the choice of the representative aK of the coset G/K. Indeed, if $aK = bK$ $(a, b \in G)$, then $b^{-1}a = k \in K$, thus $\tau_k = \tau_{b^{-1}} \circ \tau_a$. Due to the $\mathrm{Ad}^{G/K}$-invariance of $\langle\ ,\ \rangle_o$ and the commutativity of the diagram mentioned before, we have that

$$\begin{aligned}\langle d\tau_{b^{-1}}(X), d\tau_{b^{-1}}(Y)\rangle_o &= \langle d\tau_k d\tau_{a^{-1}}(X), d\tau_k d\tau_{a^{-1}}(Y)\rangle_o \\ &= \langle d\tau_{a^{-1}}(X), d\tau_{a^{-1}}(Y)\rangle_o.\end{aligned}$$

In this way we obtain a Riemannian metric on G/K, which is clearly G-invariant. For the equivalence of (2) with (3), in one direction it is a standard result of linear algebra. For the converse, we average using Haar measure to obtain an $\mathrm{Ad}^{G/K}$-invariant scalar product on $\mathfrak{g}$, and then an $\mathrm{Ad}^{G/K}$-invariant scalar product on $\mathfrak{m}$. $\square$

Remark. Proposition 5.1 is an instance of an important general phenomenon: there is a one-to-one correspondence between G-invariant objects on G/K and $\mathrm{Ad}^{G/K}$-invariant objects at $T_o(G/K) \cong \mathfrak{m}$. For instance, G-invariant tensor fields of type (p, q) correspond to $\mathrm{Ad}^{G/K}$-invariant tensors of type (p, q) on $\mathfrak{m}$.

2. The Riemannian connection

Here we will determine the Riemanian connection ∇ for a G-invariant metric g on a reductive homogeneous space $M = G/K$. For any $X \in \mathfrak{g}$, we recall the vector field $X^* \in \mathcal{X}(M)$ given by

$$X^*_o = \frac{d}{dt}(\exp tX) \cdot o \Big|_{t=0}$$

as well as the isomorphism $\mathfrak{m} \cong T_o(G/K)$. Using this vector field and the isomorphism, we may write, for the canonical projection $\pi \colon G \to G/K$, $d\pi(X) = X^*_o$, and $d\pi(X_{\mathfrak{m}}) = X^*_o$. Here $X_{\mathfrak{m}}$ denotes the component of $X \in \mathfrak{g}$ in the subspace $\mathfrak{m}$. The vector field X^* has certain special properties, namely it is a *Killing* vector field. This means that its flows ϕ_t are isometries and, equivalently, that X^* satisfies the conditions

$$(1) \qquad X^*g(Y,Z) = g([X^*,Y],Z) + g(Y,[X^*,Z])$$

and

$$(2) \qquad g(\nabla_Y X^*, Z) + g(\nabla_Z X^*, Y) = 0$$

for all $Y, Z \in \mathcal{X}(M)$. We also note that

$$(3) \qquad [X^*,Y^*] = -[X,Y]^*$$

We refer to [**ON, pp. 250-251, 255-256**] and [**Be, pp. 40-41**] for proofs of the above statements. Now, by a straightforward application of Koszul's formula and condition (1) above, we obtain that if X, Y, Z are Killing vector fields on any Riemannian manifold (M, g), then

$$(4) \qquad 2g(\nabla_X Y, Z) = g([X,Y],Z) + g([X,Z],Y) + g(X,[Y,Z]).$$

Now we can determine the Riemannian connection on $M = G/K$.

Proposition 5.2. *Let $X, Y \in \mathfrak{m}$. Then*

$$(\nabla_{X^*}Y^*)_o = -\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y),$$

where $U \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is determined by

$$(5) \qquad 2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle$$

for all $Z \in \mathfrak{m}$.

Proof. Let $X, Y, Z \in \mathfrak{m}$. Then, from property (4) and by using (3), we obtain

$$\begin{aligned} 2g(\nabla_{X^*}Y^*, Z^*) &= -g([X,Y]^*, Z^*) - g([X,Z]^*, Y^*) - g(X^*, [Y,Z]^*) \\ &= -\langle [X,Y]_{\mathfrak{m}}, Z\rangle - \langle [X,Z]_{\mathfrak{m}}, Y\rangle - \langle X, [Y,Z]_{\mathfrak{m}}\rangle \end{aligned}$$

and hence,

$$2\langle (\nabla_{X^*}Y^*)_o, Z\rangle + \langle [X,Y]_{\mathfrak{m}}, Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle$$

or

$$2\langle (\nabla_{X^*}Y^*)_o + \frac{1}{2}[X,Y]_{\mathfrak{m}}, Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle.$$

This completes the proof. □

Note. In the above proof, $(\nabla_{X^*}Y^*)_o$ lies in $T_o(G/K) \cong \mathfrak{m}$.

Definition. A homogeneous space $M = G/K$ is called *naturally reductive* if $U \equiv 0$.

Notice that the notion of naturally reductivity depends on the choice of the subspace $\mathfrak{m}$.

3. Curvature

Here we will give expressions for the sectional curvature, Ricci curvature and scalar curvature for a reductive homogeneous space $M = G/K$, following [**Be**].

Theorem 5.3. *Let $X, Y \in \mathfrak{m}$. Then the sectional curvature of $M = G/K$ is determined by the equation*

$$\begin{aligned} \langle R(X,Y)X, Y\rangle = &-\frac{3}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle - \frac{1}{2}\langle [X,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, Y\rangle \\ &- \frac{1}{2}\langle [Y,[Y,X]_{\mathfrak{m}}]_{\mathfrak{m}}, X\rangle + \langle U(X,Y), U(X,Y)\rangle \\ &- \langle U(X,X), U(Y,Y)\rangle + \langle Y, [[X,Y]_{\mathfrak{k}}, X]_{\mathfrak{m}}\rangle, \end{aligned}$$

where U is determined by equation (5).

Proof. We compute

$$
\begin{aligned}
\langle R(X,Y)X,Y\rangle &= \langle \nabla_{[X,Y]}X,Y\rangle - \langle \nabla_X\nabla_Y X,Y\rangle + \langle \nabla_Y\nabla_X X,Y\rangle \\
&= -\langle \nabla_Y X,[X,Y]\rangle - X\langle \nabla_Y X,Y\rangle + \langle \nabla_Y X,\nabla_X Y\rangle \\
&\quad + Y\langle \nabla_X X,Y\rangle - \langle \nabla_X X,\nabla_Y Y\rangle \\
&= |\nabla_Y X|^2 - \langle \nabla_X X,\nabla_Y Y\rangle + Y\langle [X,Y],X\rangle \\
&= \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 + \langle [X,Y]_{\mathfrak{m}},U(X,Y)\rangle + |U(X,Y)|^2 \\
&\quad - \langle U(X,X),U(Y,Y)\rangle + \langle [Y,[X,Y]],X\rangle \\
&\quad + \langle [X,Y],[Y,X]\rangle \\
&= |U(X,Y)|^2 - \langle U(X,X),U(Y,Y)\rangle + \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 \\
&\quad + \frac{1}{2}\langle [[X,Y]_{\mathfrak{m}},X]_{\mathfrak{m}},Y\rangle + \frac{1}{2}\langle X,[[X,Y]_{\mathfrak{m}},Y]_{\mathfrak{m}}\rangle \\
&\quad + \langle [Y,[X,Y]_{\mathfrak{k}}]_{\mathfrak{m}},X\rangle + \langle [Y,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},X\rangle - |[X,Y]_{\mathfrak{m}}|^2
\end{aligned}
$$

By using the $\mathrm{Ad}^{G/K}$-invariance of the inner product $\langle\ ,\ \rangle$, it follows that $\langle [Y,[X,Y]_{\mathfrak{k}}]_{\mathfrak{m}},X\rangle = \langle Y,[[X,Y]_{\mathfrak{k}},X]_{\mathfrak{m}}\rangle$, from which the formula is obtained. □

Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$. By polarization and scalar multiplication, the Ricci curvature at a point is determined as follows:

Proposition 5.4.

$$
\begin{aligned}
\mathrm{Ric}(X,X) &= -\frac{1}{2}\sum_i |[X,X_i]_{\mathfrak{m}}|^2 - \frac{1}{2}\sum_i \langle [X,[X,X_i]_{\mathfrak{m}}]_{\mathfrak{m}},X_i\rangle \\
&\quad - \sum_i \langle [X,[X,X_i]_{\mathfrak{k}}]_{\mathfrak{m}},X_i\rangle + \frac{1}{4}\sum_{i,j}\langle [X_i,X_j]_{\mathfrak{m}},X\rangle^2 \\
&\quad - \langle [Z,X]_{\mathfrak{m}},X\rangle,
\end{aligned}
$$

where $Z = \sum_i U(X_i,X_i)$ is determined by $\langle Z,X\rangle = \mathrm{tr}(\mathrm{ad}X)$. (Indeed, $\langle Z,X\rangle = \sum_i \langle [X,X_i]_{\mathfrak{m}},X_i\rangle = \sum_i \langle \mathrm{ad}X(X_i),X_i\rangle = \mathrm{tr}(\mathrm{ad}X)$.)

Proof. The computation is straightforward by setting $Y = X_i$ in Theorem 5.3, and then calculating $\mathrm{Ric}(X,X) = \sum_i \langle R(X,X_i)X,X_i\rangle$.
□

The above formula can be simplified slightly more by using the Killing form of $\mathfrak{g}$ ([**Be, p. 184-185**]).

Proposition 5.5.

$$\begin{aligned}\mathrm{Ric}(X,X) = &-\frac{1}{2}\sum_i |[X,X_i]_{\mathfrak{m}}|^2 - \frac{1}{2}B(X,X)\\ &+\frac{1}{4}\sum_{i,j}\langle [X_i,X_j]_{\mathfrak{m}},X\rangle^2 - \langle [Z,X]_{\mathfrak{m}},X\rangle.\end{aligned}$$

Finally, the scalar curvature is given as follows:

Proposition 5.6. *The scalar curvature of a reductive homogeneous space is given by*

$$s = -\frac{1}{2}\sum_{i,j}|[X_i,X_j]_{\mathfrak{m}}|^2 - \frac{1}{2}\sum_i B(X_i,X_i) - |Z|^2$$

For the case of a naturally reductive homogeneous space, the above formulas can be simplified drastically. For example, we have:

Proposition 5.7. *If G/K is a naturally reductive homogeneous space, then its sectional curvature is determined by the equation*

$$\langle R(X,Y)X,Y\rangle = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}},[X,Y]_{\mathfrak{m}}\rangle + \langle [[X,Y]_{\mathfrak{k}},X]_{\mathfrak{m}},Y\rangle.$$

Closing this chapter, we will mention a particularly simple class of reductive homogeneous spaces. Let G/K be a homogeneous space with G a semisimple and compact Lie group. Then every bi-invariant metric of G determines an Ad-invariant scalar product $\langle\quad\rangle$ on $\mathfrak{g}$ (cf. Proposition 3.9), hence there exists a reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$. Then the restriction of the scalar product $\langle\ ,\ \rangle$ to $\mathfrak{m}$ induces a G-invariant Riemannian metric on G/K, which is referred to as a *normal homogeneous Riemannian metric*. An important special case of a normal metric is provided by the following definition.

Definition. The normal homogeneous Riemannian metric on G/K induced by the negative of the Killing form $-B$ of $\mathfrak{g}$ is called the *standard homogeneous Riemannian metric*.

In other words, the standard metric is given by $\langle\ ,\ \rangle = (-B)|_{\mathfrak{m}}$, which is an $\mathrm{Ad}^{G/K}$-invariant inner product on $\mathfrak{m}$.

Proposition 5.8. *For a normal homogeneous space G/K the sectional curvature and Ricci curvature are given respectively as follows:*

$$\langle R(X,Y)X,Y\rangle = \langle [X,Y]_{\mathfrak{k}}, [X,Y]_{\mathfrak{k}}\rangle + \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle$$
$$\mathrm{Ric}(X,X) = -\frac{1}{4}B(X,X) + \frac{1}{2}\sum_i \langle [X,V_i]_{\mathfrak{k}}, [X,V_i]_{\mathfrak{k}}\rangle,$$

where $X, Y \in \mathfrak{m}$, and $\{V_i\}$ is a $\langle\ ,\ \rangle$-orthonormal basis in $\mathfrak{k}$.

Next, we will give two examples of computing the sectional curvature for two simple reductive homogeneous spaces. They are also naturally reductive.

Examples.

(1) Let $G/K = SO(4)/SO(3) = S^3$. The Killing form of $SO(4)$ is $B(X,Y) = 2\operatorname{tr} XY$ The subgroup $SO(3)$ is identified with the set of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

where $A \in SO(3)$. The reductive decomposition is given by $\mathfrak{o}(4) = \mathfrak{o}(3) \oplus \mathfrak{m}$, where $\mathfrak{o}(3)$ is the subalgebra of $\mathfrak{o}(4)$ consisting of all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

where $C \in \mathfrak{o}(3)$ (3×3 skew-symmetric), and $\mathfrak{m}$ is the subspace of all 4×4 matrices of the form

$$\begin{pmatrix} 0 & -x^t \\ x & 0_3 \end{pmatrix},$$

where x is a column vector in $\mathbb{R}^3$, and 0_3 the 3×3 zero matrix. We use the normal metric $\langle X, Y \rangle$ obtained by restricting $-\frac{1}{4}B = -\frac{1}{2}\operatorname{tr} XY$ on $\mathfrak{m}$. It is easy the check that the space G/K is naturally reductive, by verifying the condition

$$\langle [X,Y]_{\mathfrak{m}}, Z \rangle = \langle X, [Y,Z]_{\mathfrak{m}} \rangle$$

for all $X, Y, Z \in \mathfrak{m}$. Let e_{ij} denote the 4×4 matrix with -1 in the (i,j) entry and 1 in the (j,i) entry. Then an orthonormal basis for $\mathfrak{m}$ with respect to $\langle \ , \ \rangle$ is $\{e_{12}, e_{13}, e_{14}\}$. We compute the Lie brackets

$$[e_{12}, e_{13}] = e_{23}, \quad [e_{12}, e_{14}] = e_{24}, \quad [e_{13}, e_{14}] = e_{34},$$

from which it is evident that their restrictions to $\mathfrak{m}$ are the zero matrices. The curvature tensor is determined by the sectional curvatures of all the 2-dimensional subspaces of $\mathfrak{m}$. From Proposition 5.8 we obtain that

$$K(e_{12}, e_{13}) = \langle [e_{12}, e_{13}]_{\mathfrak{k}}, e_{23} \rangle = \langle e_{23}, e_{23} \rangle = 1.$$

Similarly, $K(e_{12}, e_{14}) = K(e_{13}, e_{14}) = 1$, hence the sectional curvature is constant 1, as expected.

(2) Let $G/K = O(4)/O(2) \times O(2)$, a Grassmann manifold of dimension 4. The subgroup K is identified with the set of matrices

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

where $A, C \in O(2)$. The reductive decomposition is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g} = \mathfrak{o}(4)$, $\mathfrak{k} = \mathfrak{o}(2) \oplus \mathfrak{o}(2)$ identified with the set of matrices of the form

$$\begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix}$$

where $D, E \in \mathfrak{o}(2)$, and $\mathfrak{m}$ consisting of the set of matrices

$$\begin{pmatrix} 0_2 & -X^t \\ X & 0_2 \end{pmatrix},$$

where X is a 2×2 real matrix. As in the previous example, we use the metric determined by restricting $-\frac{1}{4}B$. This makes the space naturally reductive. A $\langle \, , \rangle$-orthonormal basis of $\mathfrak{m}$ is the set $\{e_{13}, e_{14}, e_{23}, e_{24}\}$. A computation of the Lie brackets gives

$$[e_{13}, e_{14}] = e_{34}, \quad [e_{13}, e_{23}] = e_{12}, \quad [e_{14}, e_{24}] = e_{12}, \quad [e_{23}, e_{24}] = e_{34},$$
$$[e_{13}, e_{24}] = [e_{14}, e_{23}] = 0.$$

Their restrictions to $\mathfrak{m}$ are the zero matrices; hence by Proposition 5.8 we obtain the sectional curvatures

$$K(e_{13}, e_{14}) = K(e_{13}, e_{23}) = K(e_{14}, e_{24}) = K(e_{23}, e_{24}) = 1,$$
$$K(e_{13}, e_{24}) = K(e_{14}, e_{23}) = 0.$$

We see in this example that the sectional curvature is not constant. The above examples can be easily generalized to $SO(n+1)/SO(n) = S^n$ and $O(n+k)/O(n) \times O(k)$.

Chapter 6

Symmetric Spaces

1. Introduction

A large class of homogeneous spaces with special geometrical properties are the symmetric spaces. They were introduced by Cartan in 1925 in his attempt to classify Riemannian manifolds whose curvature tensor R satisfies the property $\nabla R = 0$. A characteristic property of a symmetric space is that every point has a global symmetry that "reverses" the geodesics through that point. Classical references in the subject are [**He**] and [**Lo**], and the more recent [**ON**], [**Be**] and [**Jos**].

Definition. A Riemannian manifold M is called *locally symmetric* if for every $p \in M$ there exists a normal neighborhood U of p such that the map

$$j_p = \exp_p \circ (-\operatorname{Id}_p) \circ \exp_p^{-1} : U \to M$$

is an isometry. Here Id_p is the identity map on T_pM.

The map j_p has the property of "reversing" the geodesics that pass through the point p. This means that if $\gamma_v : (-\epsilon, \epsilon) \to U \subset M$ is the (unique) geodesic with $\gamma_v(0) = p$ and $\gamma_v'(o) = v \in T_pM$, then $j_p(\gamma_v(t)) = \gamma_v(-t)$. Indeed, since $\gamma_v(t) = \exp_p(tv)$, we obtain that $j_p(\gamma_v(t)) = \exp \circ (-\operatorname{Id}_p)(tv) = \exp_p(-tv) = \gamma_v(-t)$.

For this reason the map j_p is called a *local geodesic symmetry* or simply a *local symmetry*. Furthermore, it is obvious that $j_p^2 = \operatorname{Id}$, and if $v \in T_pM$, then $(dj_p)_p(v) = (dj_p)_p(\gamma_v'(0)) = (j_p \circ \gamma_v)'(0) = \gamma_{-v}'(0) = -v$, hence $(dj_p)_p = -\mathrm{Id}_p$. Such an isometry is called an *involution.*

The main result of Cartan is the following:

Theorem 6.1. *A Riemannian manifold with curvature tensor R is a locally symmetric space if and only if $\nabla R = 0$.*

In particular, manifolds of constant curvature are locally symmetric.

Definition. A connected Riemannian manifold M is called a *symmetric space* if for each $p \in M$ there exists a (unique) isometry $j_p\colon M \to M$ such that $j_p(p) = p$ and $(dj_p)_p = -\mathrm{Id}_p$.

The map j_p is called a (global) *symmetry* of M at p. Equivalently, for every $p \in M$ there exists an isometry $j_p\colon M \to M$ such that $j_p^2 = \mathrm{Id}$, and p is an isolated fixed point of j_p.

Examples.

(1) The Euclidean space $\mathbb{R}^n$ is symmetric. The symmetry at $p \in \mathbb{R}^n$ is the map $j_p(x) = 2p - x$.

(2) The sphere S^n is symmetric. Since its isometry group acts transitively on S^n, it suffices to display a symmetry at the north pole $p = (1, 0, \ldots, 0)$, given by $j_p(x^1, \ldots, x^{n+1}) = (x^1, -x^2, \ldots, -x^{n+1})$.

2. The structure of a symmetric space

It is a remarkable fact that any symmetric space is actually a homogeneous space. The group of isometries is constructed by patching together local symmetries.

Theorem 6.2. *A symmetric Riemannian manifold M is homogeneous.*

Proof. We will first show that M is geodesically complete, that is, every geodesic $\gamma\colon (0, a) \to M$ is extendible. Indeed, let b be near a in $(0, a)$ and let $j_{\gamma(b)}$ be the symmetry at $\gamma(b)$. Since $j_{\gamma(b)}$ reverses geodesics through $\gamma(b)$, the required extension of γ is $j_{\gamma(b)} \circ \gamma$.

To prove that M is homogeneous, it suffices to show that for every $p, q \in M$ there exists an isometry ϕ of M that maps p to q. Let $\gamma\colon [0,1] \to M$ be a geodesic. Then the symmetry $j_{\gamma(\frac{1}{2})}$ at the point $\gamma(\frac{1}{2})$ is an isometry; call it $\tilde{\phi}$. This isometry reverses geodesics, hence carries $\gamma(0)$ to $\gamma(1)$. Since M is by definition connected, any two points $p, q \in M$ can be joined by a broken geodesic (this is a piecewise smooth curve segment whose smooth subsegments are geodesics, for example a broken geodesic in $\mathbb{R}^2$ is a polygonal curve). Thus the desired isometry ϕ that maps p to q is obtained by a finite composition of the isometries $\tilde{\phi}$ constructed above. $\square$

Since M is homogeneous, the isometry group $I(M)$ acts transitively on M, and it can be shown that the identity component $G = I_o(M)$ of $I(M)$ also acts transitively. By the Myers-Steenrod Theorem (Theorem 4.3) $I(M)$ is a Lie group, thus M can be identified with the homogeneous space G/K, where K is the isotropy subgroup of a point $p \in M$. For simplicity take $p = eK = o$, and let j denote the (global) symmetry of $M = G/K$ at o.

Next, we will see that the symmetry j provides M with further structure. We define a map $\sigma\colon M \to M$ by $\sigma(g) = j \circ g \circ j$. Clearly, $\sigma(g)$ is an isometry, hence an element of G. Since $j^2 = \mathrm{Id}$, we can write $\sigma(g) = j \circ g \circ j^{-1}$. Thus, $\sigma \quad G \to G$ is an automorphism. Let $G_\sigma = \{g \in G\colon \sigma(g) = g\}$ be the set of fixed points of σ, and G_σ^o its connected component. For the proof of the following theorem we refer to [**ON**].

Theorem 6.3. (1) *Let $M = G/K$ be a symmetric space with symmetry j at $o = eK$. Then:*

(a) *$\sigma^2 = \mathrm{Id}_G$, that is, σ is an involution.*

(b) *The set G_σ is a closed subgroup of G such that $G_\sigma^o \subset K \subset G_\sigma$.*

(These two properties make the pair (G, K) into what is called a symmetric pair.)

(2) *Conversely, if G is a connected Lie group, K a closed subgroup of G, and σ an automorphism of G satisfying (a) and (b) above, then every G-invariant metric on $M = G/K$ makes M into a Riemannian*

symmetric space such that $j \circ \pi = \pi \circ \sigma$. Here j is the symmetry of M at o, and $\pi\colon G \to M$ is the projection.

The map $\sigma\colon G \to G$ induces a map $\tilde{\sigma}\colon G/K \to G/K$ which is called the *symmetry* of G/K.

Examples.

(1) Let $G = O(n), K = O(k) \times O(n-k)$, and $\sigma\colon O(n) \to O(n)$ given by

$$\sigma(A) = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} A \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}^{-1}$$

Then $G_\sigma = K$ and the symmetric space is the Grassmann manifold $Gr_k\mathbb{R}^n$.

(2) A special case is when $G = O(n+1), K = O(n)$, and $\sigma\colon O(n+1) \to O(n+1)$ is given by

$$\sigma(A) = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}^{-1}$$

Then $G_\sigma = O(1) \times O(n), G_\sigma^o = SO(n)$ and the symmetric space is the sphere S^n.

(3) A Lie group G is a symmetric space determined by the symmetric pair $(G \times G, \Delta_G)$, where $\Delta_G = \{(g,g) \in G \times G\colon g \in G\}$ and $\sigma(g,h) = (h,g)$ $(g,h \in G)$.

Next we will see an algebraic description of a symmetric space.

Proposition 6.4. *Let G/K be a symmetric space with involution σ, and Lie algebras $\mathfrak{g}, \mathfrak{k}$ of G and K respectively. Then*

(1) $\mathfrak{k} = \{X \in \mathfrak{g}\colon d\sigma(X) = X\}$.

(2) *If $\mathfrak{m} = \{X \in \mathfrak{g}\colon d\sigma(X) = -X\}$, then $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.*

(3) *The subspace $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant, that is, $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$. Hence, a symmetric space is reductive.*

(4) *The following are true:*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Proof. (1) First, we will show that if $k \in K$, then $\sigma(k) = k$. Indeed, the differential of the isometry $\sigma(k)$ at o is $dj_o \circ dk_o \circ dj_o = dk_o$, since $dj_o = -\operatorname{Id}_o$. Then the result is obtained from the general fact that if two (local) isometries on a connected manifold have the same differentials at a point, then they coincide (see, for example, [**ON, p. 91**]). Now let $X \in \mathfrak{k}$. Since, as shown before, $\sigma|_K = \operatorname{Id}_K$, we obtain that $d\sigma(X) = X$. Conversely, let $X \in \mathfrak{g}$ with $d\sigma(X) = X$. If α is the one-parameter subgroup that corresponds to X, then the curve $\sigma \circ \alpha$ is also a one-parameter subgroup of X with the same initial velocity, hence $\sigma \circ \alpha = \alpha$. This means that $\alpha \in G_\sigma$, and in fact $\alpha \in G_\sigma^o \subset K$, thus $X \in \mathfrak{k}$.

(2) The sum is evidently direct. Now let $X \in \mathfrak{g}$ and set $X_\mathfrak{k} = \frac{1}{2}(X + d\sigma(X))$, $X_\mathfrak{m} = \frac{1}{2}(X - d\sigma(X))$. Since σ is an involution, so is $d\sigma$. Hence $d\sigma(X_\mathfrak{k}) = X_\mathfrak{k}$, so $X_\mathfrak{k} \in \mathfrak{k}$, and $d\sigma(X_\mathfrak{m}) = -X_\mathfrak{m}$, so $X_\mathfrak{m} \in \mathfrak{m}$. Thus $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

(3) Let $X \in \mathfrak{m}$ and $k \in K$. We need to show that $d\sigma(\operatorname{Ad}(k)X) = -\operatorname{Ad}(k)X$. Since $\sigma(k) = k$, the automorphisms σ and I_k (inner automorphism) of G commute. Indeed, $\sigma I_k(g) = \sigma(kgk^{-1}) = k\sigma(g)k^{-1} = I_k\sigma(g)$. Thus we have that

$$\begin{aligned} d\sigma(\operatorname{Ad}(k)X) &= d(\sigma I_k)(X) = d(I_k\sigma)(X) = \operatorname{Ad}(k)d\sigma(X) \\ &= \operatorname{Ad}(k)(-X) = -\operatorname{Ad}(k)X. \end{aligned}$$

(4) The first inclusion holds since K is a Lie subgroup of G, and the second because of (3). For the third, if $X, Y \in \mathfrak{m}$, then

$$d\sigma([X,Y]) = [d\sigma(X), d\sigma(Y)] = [-X,-Y] = [X,Y],$$

hence $[X,Y] \in \mathfrak{k}$. □

3. The geometry of a symmetric space

As we saw in Proposition 6.4, a symmetric space is reductive, and we know that the $\operatorname{Ad}(K)$-invariant subspace $\mathfrak{m}$ can be naturally identified with the tangent space $T_o(G/K)$. Furthermore, due to (4) of the same proposition ($[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$) the natural reductivity condition $\langle [X,Y]_\mathfrak{m}, Z\rangle = \langle X, [Y,Z]_\mathfrak{m}\rangle$ ($X, Y, Z \in \mathfrak{m}$) holds trivially. Here $\langle\ ,\ \rangle$

is the scalar product on $\mathfrak{m}$, corresponding to the G-invariant metric of G/K. Thus we obtain

Proposition 6.5. *Let $M = G/K$ be a symmetric space. Then*

(1) *The sectional curvature is determined by*

$$\langle R(X,Y)X,Y\rangle = \langle [[X,Y],X],Y\rangle$$

for all $X, Y \in \mathfrak{m}$.

(2) *The Ricci curvature is given by*

$$\mathrm{Ric}(X,X) = -\frac{1}{2}B(X,X), \qquad X \in \mathfrak{m}.$$

Proof. Since M is naturally reductive, the map U of Proposition 5.2 is identically zero. By taking into account the inclusions (4) of Proposition 6.4, the result is obtained by simplifying the curvature formula in Theorem 5.3. Similarly, the expression for the Ricci curvature is a direct implication of Proposition 5.5. □

4. Duality

Definition. A symmetric space is said to be of *compact type* if the Killing form B of $\mathfrak{g}$ is negative definite, and of *non-compact type* if B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{m}$.

Symmetric spaces of compact (resp. non-compact) type are also characterized by the fact that their sectional curvature is non-negative (resp. non-positive). One of the fundamental results in the theory of symmetric spaces says that every simply connected symmetric space is isometric to a product of a Euclidean space and isotropy irreducible symmetric spaces of compact or non-compact type. (Recall that a homogeneous space is called isotropy irreducible if its isotropy representation is irrecucible).

For symmetric spaces which are normal (cf. end of Chapter 5), there is a notion of duality between spaces of compact and non-compact type.

Definition. Two normal symmetric spaces $M = G/K$ and $M^* = G^*/K^*$ are called *dual* if the following are true:

(a) There exists an isomorphism of Lie algebras $\phi\colon \mathfrak{k} \to \mathfrak{k}^*$ such that $B^*(\phi(V), \phi(W)) = -B(V, W)$ for all $V, W \in \mathfrak{k}$.

(b) There exists a linear isometry $T\colon \mathfrak{m} \to \mathfrak{m}^*$ such that

$$[T(X), T(Y)] = -\phi([X, Y]) \qquad \text{for all } X, Y \in \mathfrak{m}.$$

Due to the isomorphisms $\mathfrak{m} \cong T_oM$, $\mathfrak{m}^* \cong T_oM^*$, the map T induces a linear isometry $T^*\colon T_oM \to T_oM^*$ Furthermore, dual spaces have opposite curvatures.

Example.

The dual space to the symmetric space $SU(p+q)/S(U(p)\times U(q))$ is the symmetric space $SU(p,q)/S(U(p) \times U(q))$. Here $SU(p,q)$ is the (non-compact) subgroup of $\mathrm{GL}_n\mathbb{C}$ $(n = p+q)$ that leaves the Hermitian inner product $\langle x, y\rangle = -x_1\bar{y}_1 - \quad - x_p\bar{y}_p + x_{p+1}\bar{y}_{p+1} + \quad + x_n\bar{y}_n$. Alternatively, $SU(p,q) = \{X \in \mathrm{GL}_n\mathbb{C}\colon I_{p,q}X^tI_{p,q}^{-1} = X^{-1},\ \det X = 1\}$. In particular, the dual of the sphere S^n is the hyperbolic space $\mathbb{H}^n$.

We also mention that the isotropy irreducible symmetric spaces of compact (resp. non-compact) type are divided into two categories of types I and II (resp. III and IV). We refer to [**Be, p. 195**] for their precise definitions. Hence, every simply connected symmetric space is isometric to a product of the form

$$\mathbb{R}^n \times \prod I \times \prod II \times \prod III \times \prod IV.$$

Finally, we list the isotropy irreducible symmetric spaces of compact type. For more comments on the following theorem, we refer to [**Be**], [**He**], and [**Wo2**].

Theorem 6.6. *The simply connected, isotropy irreducible symmetric spaces of compact type are the following:*

(A) *Compact simply connected groups*

$$(i)\ SU(n),\ \ (ii)\ Spin(n),\ \ (iii)\ Sp(n),\ \ (iv)\ E_6,\ \ (v)\ E_7,\ \ (vi)\ E_8,$$
$$(vii)\ F_4,\ \ (viii)\ G_2.$$

(B) *Classical spaces*

$$(ix)\ SU(p+q)/S(U(p)\times U(q)),\ \ (x)\ SO(p+q)/SO(p)\times SO(q),$$
$$(xi)\ Sp(p+q)/Sp(p)\times Sp(q),\ \ (xii) SU(n)/SO(n),$$
$$(xiii)\ SU(2n)/Sp(n),\ \ (xiv)\ SO(2n)/U(n),\ \ (xv)\ Sp(n)/U(n).$$

(C) *Exceptional spaces*

$$(xvi)\ E_6/SU(6)\times SU(2),\ \ (xvii)\ E_6/SO(10)\times SO(2),\ \ (xviii)\ E_6/F_4,$$
$$(xix)\ E_6/Sp(4),\ \ (xx)\ E_7/SU(8), (xxi)\ E_7/SO(12)\times SU(2),$$
$$(xxii)\ E_7/E_6\times SO(2),\ \ (xxiii)\ E_8/SO(16), (xxiv)\ E_8/E_7\times SU(2),$$
$$(xxv)\ F_4/Spin(9),\ \ (xxvi)\ F_4/Sp(3)\times Sp(1), (xxvii)\ G_2/SO(4).$$

The spaces (i)–$(viii)$ are of type II (the compact, connected, simply connected, and simple Lie groups), and the rest are of type I.

Chapter 7

Generalized Flag Manifolds

1. Introduction

An important class of homogeneous spaces, already mentioned in Chapter 4, is the class of generalized flag manifolds. These are homogeneous spaces of the form $G/C(S)$, where G is a compact Lie group, and $C(S)$ is the centralizer of a torus S in G. Equivalently, they are precisely the orbits of the adjoint representation of G in its Lie algebra $\mathfrak{g}$.

These homogeneous spaces have interesting geometrical properties making them useful in both differential geometry and algebraic geometry. For example, they admit a complex structure, a Kähler structure and a symplectic structure. They also admit a Kähler-Einstein metric. We will present these notions in subsequent chapters. Furthermore, they can be expressed in the form $G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is the complexification of the Lie group G and P a parabolic subgroup of $G^{\mathbb{C}}$ (cf. Section 9). In fact, they exhaust all compact, simply connected homogeneous Kähler manifolds. For this reason, they are also referred to as Kählerian C-spaces.

Generalized flag manifolds also appear in physics in a variety of contexts, e.g., as target manifolds for sigma models or as a geometric formulation of harmonic superspace [**B-F-R**]. Furthermore, they have several analogies with an important class of infinite-dimensional manifolds, the loop groups. These are spaces of maps from a circle to a Lie group (cf. [**Pr-Se**]). Generally speaking, we can say that generalized flag manifolds are a suggestive class of homogeneous spaces, where one can test conjectures.

There are many places to find more details on the topics covered in this chapter. I mention a few: [**Alek1**], [**Alek-Pe**], [**Ar1**], [**B**], [**B-H**], [**Be, Chapter 8**], [**B-F-R**], [**Gu1**], [**Gu2**], [**Koz1**], [**Koz2**], [**Nis**], [**Pi**], [**Sie**], [**Wa2, Chapter 6**], [**Wg**].

2. Generalized flag manifolds as adjoint orbits

Definition. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $w \in \mathfrak{g}$. The *adjoint orbit* of w is the set $M_w = \mathrm{Ad}(G)w = \{\mathrm{Ad}(g)w \colon g \in G\} \subset \mathfrak{g}$.

Let $K = K_w = \{g \in G \colon \mathrm{Ad}(g)w = w\}$ be the isotropy subgroup of w. Then M_w is diffeomorphic to the homogeneous space G/K. The point w corresponds to the identity coset $o = eK$.

Examples.

(1) Let $G = U(n)$ with $\mathfrak{g} = \mathfrak{u}(n)$. Let $w = \mathrm{diag}(i\lambda_1, i\lambda_2, \dots, i\lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are distinct real numbers. Then $K_w = T_n$ (an n-torus) so, $\mathrm{Ad}(U(n))w \cong F_n$, the set of all full flags in $\mathbb{C}^n$ (cf. Chapter 4, Example 10).

(2) Let $G = U(n)$ with $w = \mathrm{diag}(i\lambda_1, i\lambda_2, \dots, i\lambda_n)$, where $\lambda_1 = \quad = \lambda_k = \lambda$, $\lambda_{k+1} = \quad = \lambda_n = \mu$ $(\lambda \neq \mu)$. In this case $K_w = U(k) \times U(n-k)$ and $\mathrm{Ad}(U(n))w \cong Gr_k\mathbb{C}^n$, the Grassmann manifold of k-planes in $\mathbb{C}^n$.

(3) Let $G = SU(n)$ with $\mathfrak{g} = \mathfrak{su}(n)$. Let $w = \mathrm{diag}(i\lambda_1 I_{n_1}, i\lambda_2 I_{n_2}, \dots, i\lambda_{n_s} I_{n_s})$, where $\lambda_1, \dots, \lambda_s$ are distinct real numbers satisfying $n_1\lambda_1 + \quad + n_s\lambda_s = 0$, and where I_{n_i} is the $n_i \times n_i$ identity matrix. Then $\mathrm{Ad}(SU(n))w \cong SU(n)/S(U(n_1) \times \quad \times U(n_s))$ with $n = n_1 + \quad + n_s$. This example corresponds to the space of partial flags $F(n_1, \dots, n_s)$ in $\mathbb{C}^n$ (cf. Chapter 4, Example 11).

The homogeneous space G/K_w obtained from an adjoint orbit has a more precise expression (for a proof see [**Du-Ko**]):

Proposition 7.1. (1) *The set* $S_w = \overline{\exp \mathbb{R}w}$ *is a torus in* G.

(2) *The isotropy subgroup* K_w *is the centralizer of the torus* S_w, *that is,*

$$K_w = C(S_w) = \{g \in G \colon ghg^{-1} = h \text{ for all } h \in S_w\}.$$

(3) *If the torus* S_w *is a maximal torus in* G, *then* $C(S_w) = S_w$.

(4) *The Lie algebra of* K_w *is*

$$\mathfrak{k}_w = \{X \in \mathfrak{g} \colon [w, X] = 0\} = \ker \operatorname{ad} w.$$

Henceforth, we can give the following definition:

Definition. A *generalized flag manifold* is a homogeneous space of the form $G/K = G/C(S)$, where G is a compact Lie group and S is a torus in G. If the torus S is a maximal torus in G, say T, then G/T is called a *flag manifold.*

Proposition 7.1 enables us to give a simple description of the tangent space of M_w at w. We recall the reductive decomposition $\mathfrak{g} = \mathfrak{k}_w \oplus \mathfrak{m}_w$ of $\mathfrak{g}$ with respect to an Ad-invariant inner product on $\mathfrak{g}$ (e.g., with respect to the negative of the Killing form), where $\mathfrak{m}_w \coloneqq \mathfrak{k}_w^{\perp}$. Then

$$T_w(M_w) \cong \mathfrak{m}_w = (\ker \operatorname{ad} w)^{\perp}$$

However, due to the embedding $M_w \subset \mathfrak{g}$, there is another description of the tangent space of M_w at w:

$$\begin{aligned} T_w(M_w) &= \{\frac{d}{dt} \operatorname{Ad}(\exp tX)w|_{t=0} \quad X \in \mathfrak{g}\} \\ &= \{\frac{d}{dt} (\exp tX)w(\exp(-tX))|_{t=0} \quad X \in \mathfrak{g}\} \\ &= \{Xw - wX \colon X \in \mathfrak{g}\} = \{[X, w] \colon X \in \mathfrak{g}\} \\ &= \operatorname{Im} \operatorname{ad} w \subset \mathfrak{g}. \end{aligned}$$

Of course, these two descriptions are the same ([**Gu1**]).

3. Lie theoretic description of a generalized flag manifold

Let G/K be a generalized flag manifold. We assume that G is semisimple and compact, so the Killing form is negative definite on $\mathfrak{g}$, thus giving rise to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. As described in the previous section, K is the centralizer of a torus S in G. Let T be a maximal torus in G containing S. Then $T \subset C(S) = K$. Let $\mathfrak{h}$ be the Lie algebra of T and $\mathfrak{h}^C$ its complexification. Let R be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\alpha} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathbb{C} E_{\alpha}$$

its root space decomposition. Since $\mathfrak{k}^{\mathbb{C}}$ contains $\mathfrak{h}^{\mathbb{C}}$, there exists a subset R_K of R such that

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathbb{C} E_{\alpha}.$$

Hence we obtain

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathbb{C} E_{\alpha},$$

where $R_M = R \setminus R_K$. This is called the set of *complementary roots.* Thus we obtain that $R = R_K \cup R_M$, and finally that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{C} \oplus \mathfrak{m}^{\mathbb{C}}$$

The set $\{E_{\alpha} \colon \alpha \in R_M\}$ is a basis of the space $\mathfrak{m}^{\mathbb{C}}$. We recall that (cf. Chapter 2) the real Lie algebra $\mathfrak{g}$ is the fixed point set of the standard involution of $\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ that maps E_{α} to $-E_{-\alpha}$. Then $\{i(E_{\alpha}+E_{-\alpha}), E_{\alpha} - E_{-\alpha}\}$ span $\mathfrak{g} \cap (\mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha})$.

4. Painted Dynkin diagrams

It is possible to give a complete classification of all generalized flag manifolds with G semisimple, by the use of the painted Dynkin diagrams as follows. Let Π be a set of simple roots for the root system R. Then $\Pi_K = \Pi \cap R_K$ is a set for simple roots for the root system R_K. The pair (Π, Π_K) can be represented graphically by the *painted Dynkin diagram of type G.* This is defined as the Dynkin diagram of R, with black vertices representing the Dynkin diagram of the semisimple part of the subalgebra $\mathfrak{k}$. Hence, the white vertices represent the simple roots from $\Pi_M = \Pi \setminus \Pi_K$.

Conversely, from a painted Dynkin diagram of type G, we obtain a generalized flag manifold G/K by the following "recipe":

(a) Draw the Dynkin diagram for the semisimple algebra $\mathfrak{g}$.

(b) Paint any subset of its vertices black.

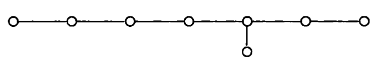
(c) The subalgebra $\mathfrak{k}$ is then obtained as the direct sum $\mathfrak{k} = \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus \mathfrak{k}'$, where each white root gives rise to one $\mathfrak{u}(1)$-summand, and the set of black roots together with the connected lines between them, yield the Dynkin diagram of $\mathfrak{k}'$, the semisimple part of $\mathfrak{k}$.

The painted Dynkin diagram determines the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{C} \oplus \mathfrak{m}^{\mathbb{C}}$ and, hence, the generalized flag manifold G/K completely up to isomorphism.

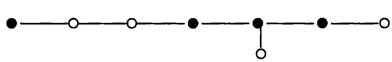
Example.

Let $\mathfrak{g} = \mathfrak{e}_8$. We exercise the three steps described above:

(a)



(b) We paint arbitrarily:



(c) Then $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$.

We obtain the generalized flag manifold

$$E_8/SU(2) \times SU(4) \times U(1)^4.$$

We can summarize the above information in the following theorem (cf. [**Alek**], [**Alek-Pe**], [**B-F-R**], [**Wg**]):

Theorem 7.2. *There is a one-to-one correspondence between generalized flag manifolds G/K of a compact semisimple Lie group G (up to isomorphism as homogeneous spaces) and painted Dynkin diagrams of type G (up to equivalence).*

There are simple rules to determine when two painted Dynkin diagrams are equivalent, but we refer to the previous references for more details.

Finally, we give the list of generalized flag manifolds of all classical Lie groups G (up to isomorphism), and the number of non-isomorphic generalized flag manifolds for the exceptional Lie groups:

$$\mathbf{A}_{n-1}\colon\ \ SU(n)/S(U(n_1)\times \qquad \times U(n_k)\times U(1)^m)$$

$$(n=\sum n_i+m,\ n_1\geq n_2\geq \qquad \geq n_k>1,\ k\geq 0,\ m\geq 0).$$

$$\mathbf{B}_n\colon\ \ SO(2n+1)/U(n_1)\times \qquad \times U(n_k)\times U(1)^m\times SO(2l+1)$$

$$\mathbf{C}_n\colon\ \ Sp(n)/U(n_1)\times \qquad \times U(n_k)\times U(1)^m\times Sp(l)$$

$$\mathbf{D}_n\colon\ \ SO(2n)/U(n_1)\times \qquad \times U(n_k)\times U(1)^m\times SO(2l)\ \ (l\neq 1)$$

$$(n=\sum n_i+m+l,\ n_1\geq n_2\geq \qquad \geq n_k>1,\ k\geq 0,\ m\geq 0, l\geq 0).$$

For the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 there are 3, 11, 16, 31, and 40 non-isomorphic generalized flag manifolds respectively.

5. T-roots and the isotropy representation

Let G/K be a generalized flag manifold with reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$. We decompose the isotropy representation $\chi=\mathrm{Ad}^{G/K}\ \ K\to \mathrm{Aut}(\mathfrak{m})$ into a direct sum

$$\chi=\chi_1\oplus\cdots\oplus\chi_s$$

of $\mathrm{Ad}^{G/K}$-invariant (or $\mathrm{Ad}(K)$-invariant from now on, for simplicity) subrepresentations χ_i. This induces a direct sum

$$\mathfrak{m}=\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_s \tag{1}$$

of $\mathfrak{m}$ into irreducible $\mathrm{ad}(\mathfrak{k})$-invariant submodules. We will now relate this decomposition with the roots $R = R_K \cup R_M$ of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$, with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ (hence $\mathfrak{k}^{\mathbb{C}}$ as well).

We define the set

$$\mathfrak{t} = \mathfrak{h}_{\mathbb{R}} \cap Z(\mathfrak{k}^{\mathbb{C}}),$$

the intersection of the real space spanned by the root system R, with the center of $\mathfrak{k}^{\mathbb{C}}$. Then $\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{t}'^{\mathbb{C}}$, where $\mathfrak{t}'^{\mathbb{C}}$ is the semisimple part of $\mathfrak{k}^{C}$ We consider the restriction map

$$\kappa\colon \mathfrak{h}^* \to \mathfrak{t}^* \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

and we set $R_T = \kappa(R) = \kappa(R_M)$ (note that $\kappa(R_K) = 0$). The elements of R_T are called *T-roots.* They were introduced in [**B-H**] and [**Sie**]. In general, R_T is not a root system. The significance of the T-roots is that they correspond to irreducible submodules of $\mathfrak{m}^{\mathbb{C}}$, with respect to the complexified isotropy representation $\chi^{\mathbb{C}}$ of G/K.

Theorem 7.3 ([**Alek-Pe**], [**Sie**]). *There exists a one-to-one correspondence between T-roots ξ and irreducible* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-invariant submodules $\mathfrak{m}^{\mathbb{C}}_{\xi}$ of $\mathfrak{m}^{C}$ This correspondence is given by*

$$R_T \ni \xi \leftrightarrow \mathfrak{m}^{\mathbb{C}}_{\xi} = \sum_{\kappa(\alpha)=\xi} \mathbb{C}E_{\alpha}.$$

As a consequence, we obtain the decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}}_{\xi_1} \oplus \cdots \oplus \mathfrak{m}^{\mathbb{C}}_{\xi_r} \tag{2}$$

of $\mathfrak{m}^{\mathbb{C}}$ into a direct sum of irreducible $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant submodules. These submodules are inequivalent, because if they were equivalent as $\mathfrak{k}^{\mathbb{C}}$-modules, then, in particular, they would have been equivalent as $\mathfrak{h}^{\mathbb{C}}$ modules, but this is impossible because the roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ are distinct, and root spaces are one-dimensional.

Furthermore, the $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant module $\mathfrak{m}^{\mathbb{C}}_{\xi} \oplus \mathfrak{m}^{\mathbb{C}}_{-\xi}$ ($\xi \in R_T$) determines an irreducible $\mathrm{ad}(\mathfrak{k})$-module $\mathfrak{m} \cap (\mathfrak{m}^{\mathbb{C}}_{\xi} \oplus \mathfrak{m}^{\mathbb{C}}_{-\xi})$, hence we obtain

$$\mathfrak{m} = \sum_{\xi \in R_T^+} (\mathfrak{m}^{\mathbb{C}}_{\xi} \oplus \mathfrak{m}^{\mathbb{C}}_{-\xi}).$$

(Here $R_T^+ = \kappa(R^+)$ is the set of all positive T-roots with respect to a fixed basis Π of R.) In other words, the precise relation between the decompositions (1) and (2) is that $r = 2s$ and that $\mathfrak{m}_i^{\mathbb{C}} = \mathfrak{m}_{\xi_i}^{\mathbb{C}} \oplus \mathfrak{m}_{-\xi_i}^{\mathbb{C}}$ $(i = 1, \ldots, s)$.

Examples.

(1) Let $M = SU(3)/S(U(1) \times U(1) \times U(1)) = G/T$. A Cartan subalgebra of the complexification of $\mathfrak{su}(3)$ has the form

$$\mathfrak{h}^{\mathbb{C}} = \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) \colon \epsilon_i \in \mathbb{C}\}.$$

The root system consists of the forms $R = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_1 - \epsilon_3), \pm(\epsilon_2 - \epsilon_3)\}$. Here $R_M = R$, and this is the set of T-roots.

The (complexified) isotropy representation has been computed in Example 3 of Section 3 in Chapter 4, and is given by

$$\mathfrak{m}^{\mathbb{C}} = K_{12} \oplus K_{13} \oplus K_{23} \oplus K_{21} \oplus K_{31} \oplus K_{32}.$$

This shows the correspondence between the set of T-roots and the $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant submodules of $\mathfrak{m}^{\mathbb{C}}$.

(2) Let $M = SO(8)/U(2) \times U(2)$. A Cartan subalgebra of the complexification of $\mathfrak{so}(8)$ has the form

$$\mathfrak{h}^{\mathbb{C}} = \{\mathrm{diag}(\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2, \epsilon_3, -\epsilon_3, \epsilon_4, -\epsilon_4) \colon \epsilon_i \in \mathbb{C}\}.$$

The root system consists of the forms $R = \{\pm(\epsilon_i \pm \epsilon_j) \colon 1 \le i \ne j \le 4\}$. Then the root system of the subalgebra $\mathfrak{k}^{\mathbb{C}}$ is $R_K = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_3 - \epsilon_4)\}$, and the complementary roots are $R_M = \{\pm(\epsilon_1 \pm \epsilon_3), \pm(\epsilon_1 \pm \epsilon_4), \pm(\epsilon_2 \pm \epsilon_3), \pm(\epsilon_2 \pm \epsilon_4), \pm(\epsilon_1 + \epsilon_2), \pm(\epsilon_3 + \epsilon_4)\}$. We choose the set of simple roots to be $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4\}$. The center of $\mathfrak{k}^{\mathbb{C}}$ as a subalgebra of $\mathfrak{h}^{\mathbb{C}}$ has the form $\{\mathrm{diag}(d_1, -d_1, d_1, -d_1, d_2, -d_2, d_2, -d_2)\}$. By restricting the complementary roots R_M to the above set, we obtain that the set of T-roots is $\{\pm(d_1 \pm d_2), \pm 2d_1, \pm 2d_2\}$.

Next, we will show the precise correspondence between the elements of the above set of T-roots and the $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant submodules of $\mathfrak{m}^{\mathbb{C}}$.

Let $\lambda_8 \colon SO(8) \to SO(8)$ and $\mu_2 \colon U(2) \to U(2)$ be the standard representations of $SO(8)$ and $U(2)$ respectively, and $\mu_2^{(i)} \colon U(2) \times U(2) \to U(2)$ the projection to the i^{th} factor ($i = 1, 2$). Then

$$\begin{aligned}
&\mathrm{Ad}^{SO(8)} \otimes \mathbb{C}\Big|_{U(2)\times U(2)} \\
&= \wedge^2(\lambda_8 \otimes \mathbb{C}|_{U(2)\times U(2)}) = \wedge^2(\mu_2^{(1)} \oplus \overline{\mu_2^{(1)}} \oplus \mu_2^{(2)} \oplus \overline{\mu_2^{(2)}}) \\
&= \wedge^2 \mu_2^{(1)} \oplus \wedge^2 \overline{\mu_2^{(1)}} \oplus \mu_2^{(1)} \ \overline{\mu_2^{(1)}} \oplus \wedge^2 \mu_2^{(2)} \oplus \wedge^2 \mu_2^{(2)} \oplus \wedge^2 \overline{\mu_2^{(2)}} \\
&\quad \oplus \mu_2^{(2)} \ \overline{\mu_2^{(2)}} \oplus \mu_2^{(1)} \ \mu_2^{(2)} \oplus \mu_2^{(1)} \ \overline{\mu_2^{(2)}} \oplus \overline{\mu_2^{(1)}} \ \mu_2^{(2)} \oplus \overline{\mu_2^{(1)}} \ \overline{\mu_2^{(2)}},
\end{aligned}$$

where we denoted by "$\cdot$" the tensor product $\otimes$. The part $\mu_2^{(1)} \cdot \overline{\mu_2^{(1)}} \oplus \mu_2^{(2)} \ \overline{\mu_2^{(2)}}$ is the complexified adjoint representation of $U(2) \times U(2)$, and the remaining part is the complexified isotropy representation $\chi^{\mathbb{C}}$ of M. By setting $\mu = \wedge^2 \mu_2^{(1)}$, $\mu' = \wedge^2 \mu_2^{(2)}$, we obtain

$$\chi^{\mathbb{C}} = \mu \oplus \overline{\mu} \oplus \mu' \oplus \overline{\mu'} \oplus (\mu_2^{(1)} \cdot \mu_2^{(2)}) \oplus (\overline{\mu_2^{(1)}} \cdot \overline{\mu_2^{(2)}}) \oplus (\mu_2^{(1)} \cdot \overline{\mu_2^{(2)}}) \oplus (\overline{\mu_2^{(1)}} \cdot \mu_2^{(2)}).$$

This induces the decomposition

$$\begin{aligned}
\mathfrak{m}^C &= \mathfrak{m}^{\mathbb{C}}_{2d_1} \oplus \mathfrak{m}^{\mathbb{C}}_{-2d_1} \oplus \mathfrak{m}^{\mathbb{C}}_{2d_2} \oplus \mathfrak{m}^{\mathbb{C}}_{-2d_2} \oplus \mathfrak{m}^{\mathbb{C}}_{d_1+d_2} \oplus \mathfrak{m}^{\mathbb{C}}_{-(d_1+d_2)} \\
&\quad \oplus \mathfrak{m}^{\mathbb{C}}_{d_1-d_2} \oplus \mathfrak{m}^{\mathbb{C}}_{-(d_1-d_2)}.
\end{aligned}$$

The correspondence with the T-roots has now been exhibited. Furthermore, we obtain the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4$ into submodules of (real) dimensions 2, 2, 8, and 8 respectively. Notice that the sum of these dimensions equals 20, the dimension of M.

6. G-invariant Riemannian metrics.

We recall that if $M = G/K$ is a homogeneous space, then according to Proposition 5.1, any G-invariant metric on M is determined by an $\mathrm{Ad}^{G/K}$-invariant scalar product on $\mathfrak{m}$. Let G be semisimple and compact, so that the Killing form on $\mathfrak{g}$ is negative definite. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be the decomposition of the isotropy representation into irreducible submodules. A G-invariant Riemannian metric is called *diagonal*, if the corresponding $\mathrm{Ad}^{G/K}$-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ can be expressed as

$$(3) \qquad \langle\ ,\ \rangle = x_1\,(-B)|_{\mathfrak{m}_1} + \cdots + x_s\,(-B)|_{\mathfrak{m}_s}$$

where $x_1, \ldots, x_s$ are positive constants.

In particular, the B-symmetric operator $A\colon \mathfrak{m} \to \mathfrak{m}$ that determines the scalar product of Proposition 5.1 is given by

$$A = x_1\,\mathrm{Id}_{\mathfrak{m}_1} + \cdots + x_s\,\mathrm{Id}_{\mathfrak{m}_s}$$

If the $\mathfrak{m}_i$'s are pairwise inequivalent representations, then the decomposition of $\mathfrak{m}$ is unique, and (3) exhausts all G-invariant metrics on G/K. Otherwise, we need not only a positive variable x_i for each irreducible submodule $\mathfrak{m}_i$, but also a parametrization of the space of all $\mathrm{Ad}^{G/K}$-equivariant maps between each pair of equivalent representations. We refer to Chapter 8, Section 5 for an example in this case.

Now, let $M = G/K$ be a generalized flag manifold. Then, as we have seen in Section 5, the submodules $\mathfrak{m}_i$ are inequivalent, hence the expression (3) describes all G-invariant metrics on M. Each such metric depends on s positive parameters $x_1, \ldots, x_s$.

Due to several advantages, it is standard practice to extend $\langle\ ,\ \rangle$ without any change in notation (a common, but dirty, trick) from $\mathfrak{m}$ to the complexification $\mathfrak{m}^{\mathbb{C}}$ by complex linearity. Hence, a G-invariant metric on M can be described by an $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant scalar product g on $\mathfrak{m}^{\mathbb{C}}$.

Let $\{\omega^\alpha\colon \alpha \in R\}$ be a vector space basis in $(\mathfrak{m}^{\mathbb{C}})^*$, which is dual to the basis $\{E_\alpha\colon \alpha \in R_M\}$ ($\omega^\alpha(E_\beta) = \delta^\alpha_\beta$). We fix a system of positive roots $R^+ = R^+_K \cap R^+_M$, and let $R^+_T = \kappa(R^+)$.

Proposition 7.4 ([**Alek-Pe**]). *Any real* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-invariant scalar product g on $\mathfrak{m}^{\mathbb{C}}$ has the form*

$$g = \sum_{\alpha \in R^+_M} g_\alpha \omega^\alpha \vee \omega^{-\alpha} = \sum_{\xi \in R^+_T} g_\xi \sum_{\alpha \in \kappa^{-1}(\xi)} \omega^\alpha \vee \omega^{-\alpha},$$

where $\omega \vee \rho = \frac{1}{2}(\omega \otimes \rho + \rho \otimes \omega)$, and the g_α are positive constants such that $g_\alpha = g_\beta$ if $\alpha|_T = \beta|_T$. Thus, a G-invariant metric on *a generalized flag manifold depends (modulo a scale factor) on* R_T^+ *parameters.*

7. G-invariant complex structures and Kähler metrics

In this section we will exploit the existence of a complex structure and a Kähler metric on a generalized flag manifold.

An *almost complex structure* on a Riemannian manifold M is a $(1,1)$-tensor J on M satisfying $J^2 = -\operatorname{Id}$, where J is thought of as a linear transformation[1] J_p on each tangent space $T_p(M)$. We denote with the same letter its extension to the complexification $T_pM^{\mathbb{C}}$ If we set

$$T_p^{(1,0)}M = \{X \in T_pM^{\mathbb{C}} \colon J_pX = iX\} \quad \text{and}$$
$$T_p^{(0,1)}M = \{X \in T_pM^{\mathbb{C}} \colon J_pX = -iX\},$$

then we obtain that $T_pM^{\mathbb{C}} = T_p^{(1,0)}M \oplus T_p^{(0,1)}M$.

An almost complex structure J is called a *complex structure* or an *integrable complex structure* if $\nabla_X J = 0$, where ∇ is the Riemannian connection of M. A complex structure means that the manifiold has coordinates that are complex-valued and with holomorphic transition functions. That is to say, they locally look like $\mathbb{C}^n$, both geometrically and analytically. If $M = G/K$ is a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and $o = eK$, then an almost complex structure is called *G-invariant* if J_o commutes with the isotropy representation of G/K; that is,

$$J_o(\operatorname{Ad}^{G/K}(k)X) = Ad^{G/K}(k)J_oX, \text{ for all } k \in \mathfrak{k}, X \in \mathfrak{m}.$$

Now let $M = G/K$ be a generalized flag manifold with root space decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}^{\alpha} \oplus \sum_{\alpha \in R_M} \mathfrak{g}^{\alpha}$. We choose a subset

[1]A $(1,1)$-tensor A can be interpreted as a linear transformation on each tangent space as follows. At the point $p \in M$, A_p can be written as a sum $\sum_i \xi_i \otimes X_i$ where $\xi_i \in T_p^*(M)$ and $X_i \in T_p(M)$. Then $A_p(Y) = \sum_i \xi_i(Y)X_i$.

R_M^+ of R_M that satisfies the conditions:

(*a*) $R = R_K \cup R_M^+ \cup R_M^-$, where $R_M^- = \{-\alpha\colon \alpha \in R_M^+\}$,

(*b*) if $\alpha \in R_K \cup R_M^+, \beta \in R_M^+$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R_M^+$.

Condition (*a*) defines an ordering in R_M (cf. Chapter 2, Section 6), and both conditions (*a*) and (*b*) define an *invariant ordering* R_M^+ in R_M.

Proposition 7.5. *There is a one-to-one correspondence between G-invariant complex structures on M and invariant orderings R_M^+ in R_M given by*

$$J_o E_{\pm\alpha} = \pm i E_\alpha \ (\alpha \in R_M^+).$$

For a proof and further discussions on G-invariant complex structures on generalized flag manifolds we refer to [**Alek-Pe**], [**B-H**], [**B-F-R**], [**Frö**], [**Nis**], [**Wg**].

Definition. A Riemannian manifold (M, g) is called *Hermitian* if it admits a complex structure J such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in T_pM$.

On a Hermitian manifold we define the *fundamental 2-form* or *Kähler form* ω by setting $\omega(X, Y) = g(JX, Y)$. This 2-form determines a bilinear form ω_p on T_pM.

Definition. A Hermitian manifold M is called *Kähler* if its fundamental 2-form is closed, that is $d\omega = 0$.

If $M = G/K$ is a homogeneous space, then, similarly with the complex structure setting, we have a notion of a G-invariant form ω determined by an $\mathrm{Ad}^{G/K}$-invariant bilinear form ω_o on $\mathfrak{m}$ or, without change in the notation, on $\mathfrak{m}^{\mathbb{C}}$

It can be shown that a G-invariant form ω on M is closed if and only if

$$\omega_o([X, Y]_{\mathfrak{m}}, Z) + \omega_o([Y, Z]_{\mathfrak{m}}, X) + \omega_o([Z, X]_{\mathfrak{m}}, Y) = 0,$$

for all $X, Y, Z \in \mathfrak{m}$. Since the Killing form B is non-degenerate on $\mathfrak{m}$, ω_o can be expressed in terms of a unique B-antisymmetric linear transformation Ω_o on $\mathfrak{m}$ such that

$$\omega_o(X,Y) = B(\Omega_o X, Y).$$

It turns out ([**B-F-R**]) that for a closed form ω_o, Ω_o must be of the form $\Omega_o = \mathrm{ad}(\gamma_o)$ ($\gamma_o \in \mathfrak{g}$), and hence we have

$$\omega_o(X,Y) = B([\gamma_o, X], Y) = B(\gamma_o, [X,Y]). \tag{4}$$

Moreover, γ_o is K-invariant, i.e., $\mathrm{Ad}(k)\gamma_o = \gamma_o$ for each $k \in K$, and infinitesimaly $\mathfrak{k} \subset \ker \mathrm{ad}(\gamma_o)$, with equality if ω is non-degenerate. In this case $\gamma_o \in Z(\mathfrak{k})$.

What is interesting, is that the adjoint orbit itself can be represented as $\mathrm{Ad}(G)\gamma_o$ ([**B-F-R, p. 614**]). Incidentally, the form ω plays a role in symplectic geometry; it is exactly the Kirillov-Kostant-Souriau form on the coadjoint orbit $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ (adjoint orbits are identified with coadjoint orbits via the Killing form of $\mathfrak{g}$).

Now, by Proposition 7.5 we obtain

$$\omega_o(E_\alpha, E_{-\alpha}) = g(J_o E_\alpha, E_{-\alpha}) = \begin{cases} ig_\alpha \text{ if } \alpha \in R_M^+, \\ -ig_\alpha \text{ if } \alpha \in R_M^-, \end{cases}$$

and

$$\omega_o(E_\alpha, E_\beta) = 0 \text{ if } \alpha + \beta \notin R.$$

On the other hand, by (4) we have that

$$\omega_o(E_\alpha, E_{-\alpha}) = B(\gamma_o, [E_\alpha, E_{-\alpha}]) = B(\gamma_o, H_\alpha) = \alpha(\gamma_o), \tag{5}$$

and so,

$$g_\alpha = \begin{cases} \frac{1}{i}\alpha(\gamma_o) \text{ if } \alpha \in R_M^+, \\ -\frac{1}{i}\alpha(\gamma_o) \text{ if } \alpha \in R_M^-; \end{cases}$$

$$g(E_\alpha, E_\beta) = 0 \text{ if } \alpha + \beta \notin R,$$

where γ_o is an appropriate element in the center of $\mathfrak{k}$. In order to have a non-degenerate metric g and closed form ω, we should have $\alpha(\gamma_o) \neq 0$ for all $\alpha \in R_M$, so γ must belong to a certain *chamber* in T Chambers in the set of T-roots are the connected components of the set $T \setminus \Gamma$, where Γ is the union of the hyperplanes $\{\xi = 0 \colon \xi \in R_T\}$ in T.

The above metric g satisfies the condition of a Kähler metric on the generalized flag manifold $M = G/K$. Hence, in view of Proposition 7.5, the Kähler structure depends upon the choice of the invariant ordering R_M^+ in R_M. We obtain:

Proposition 7.6 ([**B**]). *Let $M = G/K$ be a generalized flag manifold, and let J be a G-invariant complex structure of M corresponding to an ordering R_M^+ of R_M. Then there exist a one-to-one correspondence between chambers in T and Kähler metrics compatible with J.*

We finally give a useful criterion for a G-invariant metric to be Kähler ([**Wo-Gr**]).

Proposition 7.7. *A G-invariant metric g on M as described in Proposition 7.4 is Kähler (with respect to a choice of a G-invariant complex structure) if and only if*

$$g_\xi + g_\eta = g_{\xi+\eta} \qquad \textit{for all } \xi, \eta, \xi+\eta \in R_T^+.$$

8. G-invariant Kähler-Einstein metrics

We recall that a Riemannian manifold (M, g) is called an Einstein manifold if $\mathrm{Ric}(g) = cg$ for some constant c. Here we will see that if M is a generalized flag manifold, then for each G-invariant complex structure on M there exists a G-invariant Kähler-Einstein metric (which is essentially unique).

For a Kähler manifold (M, g, J, ω) it is convenient to introduce the Ricci 2-form ρ by $\rho(X, Y) = \mathrm{Ric}(JX, Y)$, so a Kähler metric is Einstein if and only if

$$\rho(X, Y) = c\omega(X, Y).$$

Let $M = G/K$ be a generalized flag manifold, and let us fix an invariant ordering R_M^+, hence a G-invariant complex structure J. It can be shown ([**B-H**], [**B-F-R**]) that the Ricci 2-form at the point $o = eK$, evaluated at the basis $\{E_\alpha \colon \alpha \in R_M\}$, is given by

$$\rho_o(E_\alpha, E_{-\alpha}) = 2i(\delta, \alpha),$$

where $\delta = \frac{1}{2}\sum_{\beta\in R_M^+}\beta$, and $(\ ,\ )$ is the inner product on $(\mathfrak{h}^{\mathbb{C}})^*$ induced by the Killing form on $\mathfrak{g}^{\mathbb{C}}$ It is evident now, that if we choose in (5) $\gamma_o =$ constant $\times\ \delta$, then the equation $\rho_o(E_\alpha, E_{-\alpha}) = c\omega_o(E_\alpha, E_{-\alpha})$ is satisfied. Hence, the metric g_J with

$$g_\alpha = \text{ constant } \times (\delta, \alpha),$$

is a G-invariant Kähler-Einstein metric on M, compatible with the complex structure J that corresponds to the invariant ordering R_M^+.

Proposition 7.8. *Let $M = G/K$ be a generalized flag manifold. Given a G-invariant complex structure J on M, there exists a unique G-invariant Kähler-Einstein metric g_J (up to a scalar). This is given by*

$$J \leftrightarrow R_M^+ \leftrightarrow g_J = \{g_\alpha = c\times(\delta,\alpha) \colon \delta = \frac{1}{2}\sum_{\beta\in R_M^+}\beta\}.$$

Examples.

(1) Let $M = SU(3)/S(U(1)\times U(1)\times U(1))$. The isotropy representation is decomposed into three irreducible subrepresentations (cf. Example 3, Section 3 of Chapter 4), and is given by $\mathfrak{m} = \mathfrak{m}_{12}\oplus\mathfrak{m}_{13}\oplus\mathfrak{m}_{23}$, each of (real) dimension 2. Hence, a G-invariant metric depends on three positive parameters g_{12}, g_{13}, g_{23}. A Cartan subalgbera in the complexification of $\mathfrak{su}(3)$ is given by

$$\mathfrak{h}^{\mathbb{C}} = \{\text{diag}(\epsilon_1,\epsilon_2,\epsilon_3)\colon \epsilon_i\in\mathbb{C}, \epsilon_1+\epsilon_2+\epsilon_3 = 0\}.$$

The set or roots is $R = \{\pm(\epsilon_1-\epsilon_2), \pm(\epsilon_2-\epsilon_3), \pm(\epsilon_1-\epsilon_3)\}$. Choose the ordering $R^+ = \{\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \epsilon_1-\epsilon_3\}$. So, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}\oplus\sum_{\alpha\in R}\mathbb{C}E_\alpha$, then a complex structure is given by

$$J_o(X) = \begin{cases} iX \text{ if } X\in\mathbb{C}E_\alpha,\ \alpha\in R^+, \\ -iX \text{ if } X\in\mathbb{C}E_\alpha,\ \alpha\in R^- \end{cases}$$

It is evident that there are $6 = 3!$ ways that we can select an ordering in R, hence according to Proposition 7.6 there are six $SU(3)$-invariant Kähler metrics. We find that $\delta = \sum_{\beta\in R^+}\beta = 2(\epsilon_1-\epsilon_3)$. Then, with respect to the above ordering, a Kähler-Einstein metric is given (up to scale) by

$$\begin{aligned} g_{12} &= (\delta, \alpha) = (2(\epsilon_1 - \epsilon_3), \epsilon_1 - \epsilon_2) = 2, \\ g_{13} &= (\delta, \alpha) = (2(\epsilon_1 - \epsilon_3), \epsilon_1 - \epsilon_3) = 4, \\ g_{23} &= (\delta, \alpha) = (2(\epsilon_1 - \epsilon_3), \epsilon_2 - \epsilon_3) = 2, \end{aligned}$$

so it is determined (up to scale) by the triplet $(2, 4, 2)$. There are two more Kähler-Einstein metrics $(2, 2, 4)$ and $(4, 2, 2)$ with respect to the two other pairs of complex structures.

(2) Let $M = SO(8)/U(2) \times U(2)$. We refer to Example 2 of Section 5 for the roots, T-roots, etc. The isotropy representation is decomposed into four irreducible componets $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ of dimensions $2, 2, 8$ and 8 respectively, each corresponding to a pair of T-roots $\pm 2d_1$, $\pm 2d_2$, $\pm(d_1 + d_2)$, $\pm(d_1 - d_2)$.

We define the invariant ordering $R_M^+ = \{\epsilon_1+\epsilon_3, \epsilon_1+\epsilon_4, \epsilon_2+\epsilon_3, \epsilon_2+\epsilon_4, -\epsilon_1+\epsilon_3, -\epsilon_1+\epsilon_4, -\epsilon_2+\epsilon_3, -\epsilon_2+\epsilon_4, \epsilon_1+\epsilon_2, \epsilon_3+\epsilon_4\}$, and we take the complex structure J that corresponds to this invariant ordering. Here it is more complicated to enumerate all possible invariant orderings. It turns out that there are eight invariant orderings ([**Wg**]), so eight $SO(8)$-invariant complex structures given by $J_o E_{\pm\alpha} = \pm i E_\alpha$ ($\alpha \in R_M^+$).

Since $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ (each corresponding to a pair of T-roots), an $SO(8)$-invariant metric is specified by four positive parameters g_1, g_2, g_3, g_4, where

$$\begin{aligned} g_1 &= g(E_{-\epsilon_1+\epsilon_3}, E_{\epsilon_1-\epsilon_3}) = g(E_{-\epsilon_1+\epsilon_4}, E_{\epsilon_1-\epsilon_4}), \\ &= g(E_{-\epsilon_2+\epsilon_3}, E_{\epsilon_2-\epsilon_3}) = g(E_{-\epsilon_2+\epsilon_4}, E_{\epsilon_2-\epsilon_4}), \\ g_2 &= g(E_{\epsilon_1+\epsilon_3}, E_{-(\epsilon_1+\epsilon_3)}) = g(E_{\epsilon_1+\epsilon_4}, E_{-(\epsilon_1+\epsilon_4)}) \\ &= g(E_{\epsilon_2+\epsilon_4}, E_{-(\epsilon_2+\epsilon_4)}), \\ g_3 &= g(E_{\epsilon_1+\epsilon_2}, E_{-(\epsilon_1+\epsilon_2)}) \\ g_4 &= g(E_{\epsilon_3+\epsilon_4}, E_{-(\epsilon_3+\epsilon_4)}). \end{aligned}$$

We find that $\delta = \frac{1}{2}\sum_{\beta \in R_M^+} \beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + 5\epsilon_3 + 5\epsilon_4)$. Thus the Kähler-Einstein metric compatible with the complex structure J is given by

$$g_1 = (-\epsilon_1 + \epsilon_3, \delta) = 2, \qquad g_2 = (\epsilon_1 + \epsilon_3, \delta) = 3,$$
$$g_3 = (\epsilon_1 + \epsilon_2, \delta) = 1, \qquad g_4 = (\epsilon_3 + \epsilon_4, \delta) = 5,$$

so it is determined (up to scale) by the ordered numbers $(2, 1, 3, 5)$.

In both examples it is easy to see that the components of the G-invariant metric g satisfy the relations $g_\xi + g_\eta = g_{\xi+\eta}$ $(\xi, \eta, \xi+\eta \in R_T^+)$ of Proposition 7.7.

9. Generalized flag manifolds as complex manifolds

A generalized flag manifold G/K can be identified with $G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is the complexification of the Lie group G, and P is a *parabolic subgroup* of $G^{\mathbb{C}}$. This is a Lie subgroup of $G^{\mathbb{C}}$ containing a *Borel subgroup*, i.e., a maximal solvable subgroup. We see this as follows:

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in R} \mathfrak{g}^\alpha$ be the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ We fix a basis Π for R, and let R^+ be the set of positive roots with respect to Π. Then the subset

$$\mathfrak{b} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R^+} \mathfrak{g}^\alpha$$

is a *Borel subalgebra* (i.e., a maximal solvable subalgebra) of $\mathfrak{g}^{\mathbb{C}}$.

Example.

If $G = SU(n)$, then $G^{\mathbb{C}} = \mathfrak{sl}_n\mathbb{C}$ and $\mathfrak{b}$ consists of the upper triangular matrices in $\mathfrak{sl}_n\mathbb{C}$.

The Borel subalgebra $\mathfrak{b}$ defined above is called the *standard* Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Every Borel subalgebra is conjugate to this ([**Hu**]). A parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ is one that contains a Borel subalgebra. They can be constructed as follows:

Let Π_K be a subset Π, and let

$$R_K = \operatorname{span} \Pi_K \cap R,$$

so that $\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}^{\alpha}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $R_M^+ = R^+ \backslash R_K$ and take $\mathfrak{u} = \sum_{\alpha \in R_M^+} \mathfrak{g}^{\alpha}$. Then, the subalgebra

$$\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{u}$$

contains $\mathfrak{h}^{\mathbb{C}}$, and is hence parabolic.

Finally, we mention that the generalized flag manifolds exhaust all compact, simply connected homogeneous Kähler manifolds as was shown in [**B**] and [**Wg**]. For this reason, they are also (occasionally) referred to in the literature as *Kählerian C-spaces.*

Chapter 8

Advanced topics

1. Einstein metrics on homogeneous spaces

We have already mentioned that a Riemannian manifold (M, g) is called an *Einstein manifold* if the Ricci curvature of g satisfies the equation $\mathrm{Ric}(g) = cg$ for some constant c. Einstein metrics are generally considered as privileged metrics on a given Riemannian manifold. There are several reasons that justify this statement, and the simplest is the one that appears in the very first pages in the book by Besse ([**Be, pp. 1-5**]).

If a manifold has dimension 2 (a classical surface), a privileged metric can be considered as one with constant Gauss curvature. In an attempt to generalize this to an arbitrary Riemannian manifold, we have the option to impose constancy to the three notions of curvature that exist there: the sectional curvature, the Ricci curvature, and the scalar curvature. Constancy of the sectional curvature is too strong a condition, as in this case a simply connected Riemannian manifold of dimension n (other than 3) is locally isometric to either the sphere S^n, the Euclidean space $\mathbb{R}^n$, or the hyperbolic space $\mathbb{H}^n$. On the other hand, constancy of the scalar curvature turns out to be too weak a condition, as there are infinite families of Riemannian metrics on a given manifold that satisfy this property. Hence, we are left to impose constancy of the Ricci curvature, which reduces to the equation $\mathrm{Ric}(g) = cg$.

Another reason that Einstein metrics are considered to be privileged metrics, is that they appear as critical points of the total *scalar curvature functional* $g \mapsto \int_M \mathrm{S}_g \, d\,\mathrm{vol}_g$, defined on the set of all Riemannian metrics of volume 1 (cf. [**Ber2**], [**Ber3**], [**Mu**]).

Finally, Einstein metrics are related to general relativity. Indeed, Einstein had proposed the field equations for the interaction between gravity and space-time, as $\mathrm{Ric}(g) - \frac{1}{2}S_g g - \lambda g = T$, where T is the energy-momentum tensor, and λ the cosmological constant. In the vacuum ($T = 0$), the field equations reduce to $\mathrm{Ric}(g) = cg$.

The general question of classification of all homogeneous spaces that admit an Einstein metric, as well as the complete description of all G-invariant Einstein metrics on a given Riemannian homogeneous space $(M = G/K, g)$, is still open. The bibliography on the subject is vast, and I will not make any attempt to present it. The book by Besse (it contains results up to 1986) is certainly a good start, as well as the article of W. Ziller in [**Zi**]. More recent results are collected in [**L-Wa**]. In this section we will present a few results about Einstein metrics on homogeneous spaces that have been mentioned so far in this book.

Isotropy irreducible spaces.

Let $M = G/K$ be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and isotropy representation $\mathrm{Ad}^{G/K}\ K \to \mathrm{Aut}(\mathfrak{m})$. We assume that G is compact. Then we have the following:

Theorem 8.1 (J. A. Wolf). *If $M = G/K$ is an isotropy irreducible homogeneous space, then M admits a unique (up to homotheties) G-invariant Riemannian metric. This metric is Einstein.*

Proof. Since G is compact, it admits an Ad-invariant scalar product (e.g., minus the Killing form) which, by restriction to the subspace $\mathfrak{m}$, induces a G-invariant metric on M (recall that for the case of the Killing form, this metric is called standard, cf. Chapter 5). Let g, g' be two such metrics. Since the isotropy representation is irreducible, the subspace $\mathfrak{m}$ cannot be decomposed into a direct sum of irreducible $\mathrm{Ad}^{G/K}$-modules, hence the two metrics $g|_o$ and $g'|_o$ are proportional.

In particular, the Ricci tensor Ric at the point $o = eK$ is an $\mathrm{Ad}^{G/K}$-invariant symmetric bilinear form, thus $\mathrm{Ric} = cg$ at the point o, and by the invariance at any other point, hence g is Einstein.

In [**Wo1**] Wolf classified all isotropy irreducible homogeneous spaces which are not symmetric, assuming that the identity component of K acts irreducibly on $\mathfrak{m}$. Symmetric spaces are isotropy irreducible, hence Einstein manifolds. For further results about Einstein metrics on symmetric spaces we refer to [**Ke1**] and its references.

Normal homogeneous spaces.

Recall that a normal homogeneous space G/K is a Riemannian homogeneous space with a G-invariant metric, which is induced from a bi-invariant metric on G. In [**Wa-Zi1**] M. Wang and W. Ziller classified all homogeneous spaces, with G simple, for which the normal metric is Einstein. Their classification is based on the following procedure.

We first need the notion of the Casimir operator of a representation of a Lie algebra. Let $\rho\colon \mathfrak{g} \to \mathrm{Gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ an ideal of $\mathfrak{g}$ with $\dim \mathfrak{h} = n$. We assume that the bilinear form $\tau(X,Y) = \mathrm{tr}(\rho(X)\circ\rho(Y))$ $(X, Y \in \mathfrak{g})$ is non-degenerate when restricted to $\mathfrak{h}\times\mathfrak{h}$. Let $\{X_i\}, \{X_i'\}$ be dual bases of $\mathfrak{h}$ (that is, $\tau(X_i, X_j') = \delta_{ij}$). Then the element

$$C_{\rho,\tau} = \sum_{i=1}^{n} \rho(X_i)\circ\rho(X_i')$$

is an endomorphism of V that commutes with every endomorphism $\rho(A)$ $(A \in \mathfrak{g})$, and is called the *Casimir operator* (or Casimir element) of ρ. It can be shown that $\mathrm{tr}\, C = n$.

If we denote by χ the isotropy representation of G/K, then the Ricci curvature of G/K can be expressed by using the Casimir operator of χ.

Theorem 8.2 (M. Wang and W. Ziller). *If G/K is a normal homogeneous space, then*

$$\mathrm{Ric}(X,X) = -\frac{1}{4}B(X,X) + \frac{1}{2}B(C_{\chi,-B}X, X).$$

Corollary 8.3. *A normal homogeneous space is Einstein if and only if* $C_{\chi,-B} = c\,\mathrm{Id}$ *for some constant* c.

From the above corollary we see that the classification reduces to the computation of the Casimir operator of the isotropy representation of G/K. This is done for the case that G is simple. The semisimple case has been studied by E. D. Rodionov in [**Ro1**], [**Ro2**].

Einstein metrics on generalized flag manifolds.

The generalized flag manifolds have been presented in the previous chapter. They are orbits of the adjoint representation of a compact Lie group G, and equivalently, they are homogeneous spaces of the form $G/C(S)$, where S is a torus in G. As we saw in Sections 7 and 8 of Chapter 7, generalized flag manifolds admit a finite number of G-invariant complex structures, and there is a one-to-one correspondence between complex structures (up to sign) and G-invariant Kähler-Einstein metrics (up to a constant factor). Furthermore, for some of these spaces the standard metric is Einstein, since they appear in the list of Wang and Ziller ([**Wa-Zi1**]) of normal homogeneous spaces.

The question of finding other G-invariant Einstein metrics on generalized flag manifolds has been studied by D.V Alekseevsky, M. Kimura, Y. Sakane, and the author in [**Alek2**], [**Ki**], [**Sak**], and [**Ar1**] respectively. Kimura used the variational approach to find G-invariant Einstein metrics as described in [**Wa-Zi2**]. It is remarkable that the Einstein equation for a generalized flag manifold reduces to a non-linear algebraic system of equations. In fact, this is only a special case of how Lie theory can be applied to transfer problems from analysis to algebra. For example, the Ricci curvature of a G-invariant metric on a generalized flag manifold as described in Proposition 7.4, can be expressed in Lie terms as shown in [**Ar1**]. By using this expression of the Ricci curvature it is possible to give solutions for certain large families of generalized flag.

Theorem 8.4 ([**Ar1**], [**Ki**]). *For the space $SU(n)/S(U(n_1)\times U(n_2)\times U(n_3))$ $(n = n_1+n_2+n_3)$ the Einstein equation reduces to the system*

$$n_i + n_j + \frac{1}{2}\sum_{k\neq i,j} \frac{n_k}{g_{ik}g_{jk}}(g_{ij}^2 - (g_{ik} - g_{jk})^2) = g_{ij},$$

of three equations with three unknowns g_{12}, g_{13}, g_{23}. The solution of this system is the following:

(*a*) $g_{12} = n_1 + n_2,\quad g_{13} = n_1 + 2n_2 + n_3,\quad g_{23} = n_2 + n_3;$

(*b*) $g_{12} = n_1 + n_2 + 2n_3,\quad g_{13} = n_1 + n_3,\quad g_{23} = n_2 + n_3;$

(*c*) $g_{12} = n_1 + n_2,\quad g_{13} = n_1 + n_3,\quad g_{23} = 2n_1 + n_2 + n_3;$

(*d*) $g_{12} = n_1 + n_2,\quad g_{13} = n_1 + n_3,\quad g_{23} = n_2 + n_3.$

Hence it admits (up to scale) precisely four $SU(n)$-invariant Einstein metrics. The metrics (a)-(c) are Kähler metrics and the metric (d) is non-Kähler. If $n_1 = n_2 = n_3$, the metric (d) is the standard metric.

Theorem 8.5 ([**Ar1**]). *For the space $SU(n)/S(U(1)\times \quad \times U(1))$ (n times), the Einstein equation reduces to the system*

$$2 + \frac{1}{2}\sum_{k\neq i,j} \frac{1}{g_{ik}g_{jk}}(g_{ij}^2 - (g_{ik} - g_{jk})^2) = g_{ij}$$

of $\frac{1}{2}n(n-1)$ equations with $\frac{1}{2}n(n-1)$ unknowns g_{ij} $(1\leq i<j\leq n)$. For $n = 3$, the system admits the four solutions as a special case of Theorem 8.4. For $n \geq 4$, it admits at least $\frac{n!}{2}+1+n$ solutions (hence Einstein metrics up to scale). The $n!/2$ metrics are Kähler-Einstein metrics, one is the standard metric, and the remaining n Einstein metrics are given explicitly as follows:

$$g_{si} = g_{sj} = n-1 \quad (i\neq s, j\neq s),$$
$$g_{kl} = n+1 \quad (k,l\neq s),$$
$$\text{for each} \quad (1\leq s\leq n).$$

Example. Apply the previous theorem for the flag manifold $M = SU(3)/S(U(1)\times U(1)\times U(1))$. We know that (cf. Example 4, Section 3, Chapter 4) the isotropy representation is decomposed as $\mathfrak{m} = \mathfrak{m}_{12}\oplus$

$\mathfrak{m}_{13} \oplus \mathfrak{m}_{23}$, hence a $SU(3)$-invariant metric depends on three positive numbers g_{12}, g_{13}, and g_{23}. The system in Theorem 8.5 reduces to

$$
\begin{aligned}
g_{12}^2 - g_{13}^2 - g_{23}^2 + 6g_{13}g_{23} &= 2g_{12}g_{13}g_{23}, \\
g_{13}^2 - g_{12}^2 - g_{23}^2 + 6g_{12}g_{23} &= 2g_{12}g_{13}g_{23}, \\
g_{23}^2 - g_{12}^2 - g_{13}^2 + 6g_{12}g_{13} &= 2g_{12}g_{13}g_{23}.
\end{aligned}
$$

The four solutions (up to scale) are given by: (a) $g_{12} = g_{13} = g_{23}$, (b) $g_{12} = g_{23}$, $g_{13} = 2g_{12}$, (c) $g_{13} = g_{12}$, $g_{23} = 2g_{13}$, (d) $g_{13} = g_{23}$, $g_{12} = 2g_{13}$. Solution (a) corresponds to the standard metric, and solutions (b)–(d) are Kähler-Einstein metrics.

An improved version of Theorem 8.5 was obtained by Y. Sakane in [**Sak**] by the use of Gröbner bases.[1] In fact, he gave a new class of Einstein metrics when $n = 2m$ $(m > 2)$, and a complete solution for $SU(4)/S(U(1) \times U(1) \times U(1) \times U(1))$. It is possible to find Einstein metrics for several other examples of generalized flag manifolds, especially if the isotropy representation decomposes into three or four irreducible components, so that there are not too many unknowns that determine the G-invariant metric. We refer to [**Ki**] and [**Ar1**] for more such examples.

2. Homogeneous spaces in symplectic geometry

A symplectic manifold is a manifold of even-dimension equipped with a 2-form ω that is closed, and non-degenerate. We have seen examples of homogeneous spaces that are symplectic manifolds, namely the generalized flag manifolds, viewed as adjoint orbits of a Lie group. In Chapter 7 we have discussed several aspects about the geometry of generalized flag manifolds; here we will consider an application related to the existence of the symplectic structure, which will be a Hamiltonian system. This consists of a symplectic manifold M, a function $H\colon M \to \mathbb{R}$, and a differential equation of the form $\dot{x} = V_x^H$, where V^H is a vector field such that $\omega(V^H, \ ) = dH$. A classical such example, as we will see, are Newton's equations for the motion of a

[1] This is a basis on a polynomial ideal generated by a set of polynomials (a notion often used in algebraic geometry). One of the uses of Gröbner bases is to solve algebraic systems of equations.

particle in a field with potential V In this section we will give an important example of a Hamiltonian system on an adjoint orbit.

Hamiltonian systems are special examples of integrable systems. An integrable system is a certain differential equation which has a special algebraic and geometric significance. For a survey of integrable systems from the point of view of Lie groups and Lie algebras we refer to the book by A. Perelomov ([**Pe**]), and to several articles in [**At2**]. For integrable systems related to harmonic maps and loop groups, as well as for recent developments we refer to the book by M. Guest [**Gu1**]. Our approach follows closely this book.

A classical Hamiltonian system.

Definition. A *symplectic manifold* is a manifold M equipped with a 2-form ω which is closed ($d\omega$=0) and non-degenerate (i.e., if $\omega(X,Y) = 0$ for all $Y \in T_pM$, then $X = 0$).

The form ω is called a *symplectic form* on M.

Let $H\colon M \to \mathbb{R}$ be a function on a symplectic manifold M. Then a *Hamiltonian system* on M is an equation of the form

$$\dot{x} = V^H_x, \tag{1}$$

where $x\colon \mathbb{R} \to M$ is a path in M, and V^H is a vector field on M such that $\omega(V^H, \cdot\,) = dH$. The function H is called the *Hamiltonian function* of the system, and V^H is called the *Hamiltonian vector field* of the system. The terminology has its origin in the classical formulation of Hamiltonian mechanics (see for example [**Arn**]).

A classical Hamiltonian system is obtained from Newton's equations

$$m\ddot{q}_j = -\frac{\partial V}{\partial q_j}$$

for the motion of a particle of mass m in a field with potential $V = V(q)$, which is a function of the position $q = (q_1, \ldots, q_n)$ of the particle in $\mathbb{R}^n$. If we define the *momentum* and *total energy* to be $p_j = m\dot{q}_j$ ($j = 1, \ldots, n$) and $H(p,q) = \frac{1}{2m}\sum_{j=1}^n p_j^2 + V(q)$ respectively, then we obtain Hamilton's equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \tag{2}$$

Let $\operatorname{grad} H\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the *gradient* of H defined as

$$\langle(\operatorname{grad} H)_p, q\rangle = dH_p(q)$$

for all $p, q \in \mathbb{R}^n \times \mathbb{R}^n$, with respect to the standard inner product on $\mathbb{R}^n \times \mathbb{R}^n$. If we set

$$x = \begin{pmatrix} p^t \\ q^t \end{pmatrix}, \text{ and } J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

then equations (2) become

$$\dot{x} = J(\operatorname{grad} H)_x.$$

This equation is of the form (1) with $V^H = J \operatorname{grad} H$, provided we define $\omega(X,Y) = \langle JX, Y\rangle$. A short computation shows that ω satisfies $d\omega = 0$. More generally, the above procedure can be applied whenever we have a Riemannian manifold (M, g) with a complex structure J, and a 2-form ω defined by $\omega(X,Y) = g(JX,Y)$, and satisfying $d\omega = 0$. The next example will use these ingredients to define a Hamiltonian system.

A Hamiltonian system on generalized flag manifolds.

We will discuss an example of a Hamiltonian system on a generalized flag manifold $M_w = \operatorname{Ad}(G)w = G/K$, viewed as an adjoint orbit for some $w \in \mathfrak{g}$. Notice the embedding $M_w \subset \mathfrak{g}$. We need to define a Hamiltonian function and a symplectic form on M_w. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the reductive decomposition with respect to an Ad^G-invariant inner product $(\ ,\ )$ on $\mathfrak{g}$ (as usual, if G is compact, we can take this to be the negative of the Killing form on G).

Let $Q \in \mathfrak{g}$. Then define $H^Q\colon M_w \to \mathbb{R}$ by the formula

$$H^Q(\operatorname{Ad}(g)w) = (\operatorname{Ad}(g)w, Q).$$

This is called a *height function*, or a *projection Hamiltonian* on the adjoint orbit M_w. (This is related to another natural function that could have been considered, the *distance function* $K^Q(\operatorname{Ad}(g)w) =$

$|\mathrm{Ad}(g)w - Q|^2$. Since $K^Q = a + bH^Q$, for some constants a, b, it is enough to study the first function.

Next, we need to define the symplectic form ω. This will be a G-invariant 2-form, hence it will correspond to an $\mathrm{Ad}^{G/K}$-invariant bilinear form ω_o on $\mathfrak{m}$. We define this as

$$\omega_o(X, Y) = (w, [X, Y]) \qquad (X, Y \in \mathfrak{m}),$$

and compare this with (4) of Section 7 in Chapter 7. In that section we saw that ω is closed, and it is easy to see that it is non-degenerate. Hence, we have obtained a Hamiltonian system

$$\dot{x} = V_x^H \tag{3}$$

on the adjoint orbit $\mathrm{Ad}(G)W$

We recall that we had also defined a complex structure J on the adjoint orbit, by $J_o E_{\pm\alpha} = \pm i E_\alpha$, for each $\alpha \in R_M^+$, the set of complementary roots. This complex structure satisfies the property $\omega_o(X, Y) = g(J_o E_\alpha, E_{-\alpha})$, with respect to a G-invariant metric, as defined in Proposition 7.4. Now, we can see that, with this choice of a complex structure and G-invariant metric on M_w, we have that

$$V^H = J \operatorname{grad} H^Q$$

Indeed, it is enough to verify that such a V^H satisfies $\omega_o(V^H, X) = dH^Q(X)$ for all $X \in \mathfrak{m}^{\mathbb{C}}$. We simply need to check this on the basis $\{E_\alpha \colon \alpha \in R_M\}$. Let $\alpha \in R_M^+$ (similarly for $\alpha \in R_M^-$). By definition of the gradient vector field $\operatorname{grad} H^Q$,

$$dH^Q(X) = g(X, \operatorname{grad} H^Q).$$

Hence, we need to verify the equality

$$\omega_o(J_o \operatorname{grad} H^Q, E_\alpha) = g(E_\alpha, \operatorname{grad} H^Q),$$

for each $\alpha \in R_M^+$. If $\operatorname{grad} H^Q \in \mathbb{C}E_\alpha$ $(\alpha \in R_M^+)$, both sides vanish. If $\operatorname{grad} H^Q \in \mathbb{C}E_\alpha$ $(\alpha \in R_M^-)$, then they are both equal to a multiple of g_a, hence we are done. Thus the Hamiltonian system (3) reduces to

$$\dot{x} = J(\operatorname{grad} H^Q)_x. \tag{4}$$

Also, because of the relation $\omega_o(X,Y) = g(J_oX,Y)$, we obtain that $\operatorname{grad} H^Q = JV^H$, and it turns out (cf. **[Pi, (4.7), (4.10)]**) that $V^H = Q^*$ Thus (4) reduces further to

$$\dot{x} = -Q^*_x. \tag{5}$$

Now, because of the embedding $M_w \subset \mathfrak{g}$, we have that

$$\begin{aligned} Q^*_{x=\mathrm{Ad}(g)w} &= \frac{d}{dt}\,\mathrm{Ad}(\exp tQ)\mathrm{Ad}(g)w|_{t=0} \\ &= \frac{d}{dt}\,(\exp tQ)\mathrm{Ad}(g)w(\exp(-tQ))|_{t=0} = [Q,x], \end{aligned}$$

and thus, equation (5) finally reduces to

$$\dot{x} = [x,Q],$$

which is a differential equation for $x\colon \mathbb{R} \to M_w$.

Proposition 8.6. *The (unique) solution to the differential equation*

$$\dot{x} = [x,Q], \quad x(0) = w$$

is given by $x(t) = \mathrm{Ad}(\exp(-tQ))w$.

Proof. We check that this is indeed a solution:

$$\begin{aligned} \dot{x}(t) &= \frac{d}{ds}\,x(t+s)|_{t=0} = \frac{d}{ds}\,\mathrm{Ad}(\exp(-(t+s))Q)w|_{t=0} \\ &= \mathrm{Ad}(\exp(-tQ))\,\frac{d}{ds}\mathrm{Ad}(\exp(-sQ))w\Big|_{t=0} \\ &= \mathrm{Ad}(\exp(-tQ))[-Q,w] = \mathrm{Ad}(\exp(-tQ))[w,Q] \\ &= [\mathrm{Ad}(\exp(-tQ))W,Q] = [x(t),Q]. \quad \square \end{aligned}$$

The above differential equation is of the form

$$\dot{L} = [L,M]$$

which is called a *Lax equation.* Such equations are important in the theory of integrable systems. For example, Hamiltonian systems for the Toda Lattice (that is, systems which describe the motion of particles moving in a straight line, with "exponential interactions"), are equivalent to a Lax equation. We refer to the recent book by M. Guest (**[Gu1]**) and its references, for more details.

3. Homogeneous geodesics in homogeneous spaces

Geodesics in a Riemannian manifold generalize the notion of a straight line in a Euclidean space, being curves that minimize length. In fact, a fundamental theorem in Riemannian geometry says the following:

Theorem 8.7 (Hopf-Rinow). *Let (M, g) be a connected Riemannian manifold. We define the distance between any two points p and q in M as the minimum length (with respect to g) of all curves from p to q. This distance makes M into a metric space, and we assume that it is complete (i.e., every Cauchy sequence converges). Then, any two points p, q in M can be joined by a geodesic whose length is the distance from p to q.*

Besides their importance in geometry, geodesics also have applications in mechanics. More specifically, the motion of a rigid body along geodesics in the group $SO(3)$ of rotations of the three-dimensional Euclidean space equipped with a left-invariant metric, has special importance; it is called *Euclidean motion* of a rigid body ([**Arn, pp. 318–323**]). Such an example is the motion of a body in an ideal (incompressible and inviscid) fluid.

If G is a Lie group with a bi-invariant metric, then we have seen in Chapter 3, Section 3 that geodesics in G through the point e are the one-parameter subgroups of G, that is, curves of the form $\gamma(t) = \exp tX$ $(X \in \mathfrak{g})$. If the metric is simply left-invariant, then a geodesic is a one-parameter subgroup if and only if $(\mathrm{ad}_X)^* X = 0$ for all $X \in \mathfrak{g}$ (cf. [**Ch-Eb, p. 64**]). Furthermore, V V. Kajker in [**Kaj**] showed that a connected Lie group has at least one homogeneous geodesic (the term to be defined later on) through the identity element e, and J. Szenthe in [**Sz**] proved that if the Lie group is compact, semisimple and of rank ≥ 2, then there are infinitely many homogeneous geodesics through e. In [**Ma**] R. A. Marinosci has obtained more results in this direction. In mechanics, such geodesics are called *stationary rotations* (cf. [**Arn, p. 328**]). These are rotations of a rigid body for which the angular velocity is constant.

If $M = G/K$ is a Riemannian homogeneous space, then a geodesic through a point $p \in M$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of isometries of M. Hence it is of the form $\gamma(t) = (\exp tX) \cdot p$ ($X \in \mathfrak{g} \setminus \{0\}$). Such geodesics have been studied by several authors. In [**Kos**] and [**Vi**], B. Kostant and E. B. Vinberg found a simple condition that the orbit $\gamma(t) = \exp tX \cdot o$ through the point $o = eK \in G/K$ is a geodesic. Homogeneous geodesics also have important applications in mechanics. For example, the equation of motion of many systems in classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold. Such geodesics are called by V. A. Arnold *relative equilibriums* (cf. [**Arn, p. 379**].

From the above we see that Riemannian homogeneous spaces such that all geodesics are homogeneous, are of special importance. They are usually known as *g.o. spaces.* Examples of such spaces are the naturally reductive homogeneous spaces. For some time it was incorrectly believed (e.g., [**Am-Si, Theorem 5.4**]), that these are the only spaces such that all geodesics are orbits. However, in [**Ka**] A. Kaplan gave examples of g.o. spaces which are in no way naturally reductive.

A systematic study of g.o. spaces was initiated by O. Kowalski and L. Vanhecke in [**Kow-Va**], who classified all g.o. spaces of dimension ≤ 6. Later on, C. Gordon in [**Go**] described g.o. spaces M that are nilmanifolds (i.e., a nilpotent Lie group with a left-invariant metric). Also, in [**Kow-Sz**], O. Kowalski and J. Szenthe showed that every homogeneous Riemannian manifold admits at least one homogeneous geodesic through each point. Another approach for the description of g.o. spaces by using the notion of a geodesic graph was proposed by O. Kowalski and S. Nikĕvić in [**Kow-Ni**] and Z. Dušek in [**Dus1**], [**Dus2**]. An explicit description of homogeneous geodesics in certain examples, was recently given in [**Kow-Ni-Vl**]. Finally, in [**Alek-Ar**] D. V. Alekseevsky and the author initiated a study of G-invariant metrics on generalized flag manifolds, under which these are g.o. spaces.

Homogeneous spaces all of whose geodesics are orbits (g.o. spaces).

Let M be a Riemannian homogeneous space, that is a homogeneous space $M = G/K$ with a G-invariant metric g. Assume for simplicity that G is compact, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition (for example, with respect to the negative of the Killing form B of G). Recall that the G-invariant metric determines an $\mathrm{Ad}^{G/K}$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$, and an $\mathrm{Ad}^{G/K}$-invariant B-symmetric operator A on $\mathfrak{m}$.

We also recall that a homogeneous space G/K is called naturally reductive (cf. definition after Proposition 5.2) if there exists a reductive decomposition satisfying

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle [X,Z]_{\mathfrak{m}}, Y\rangle = 0 \qquad \text{for all } X,Y,Z \in \mathfrak{m}.$$

An important characteristic property of naturally reductive homogeneous spaces is given in the next proposition. Its proof uses results about Riemannian submersions that we have not developed in this book. We refer to [**ON**] or [**Ch-Eb**] for a proof.

Proposition 8.8. *Let $M = G/K$ be a naturally reductive homogeneous space. Then, each geodesic of M starting at $o = eK$ is given by*

$$\gamma(t) = (\exp tX)\cdot o, \ X \in \mathfrak{m}.$$

We have seen in Chapter 6 that symmetric spaces are naturally reductive, hence their geodesics are of the above form.

The existence of homogeneous geodesics in any Riemannian homogeneous space is guaranteed by the following theorem:

Theorem 8.9 ([**Kow-Sz**]). *Every Riemannian homogeneous space $(M = G/K, g)$ admits at least one homogenous geodesic through each point $o \in M$. If, in addition, the group G is semisimple, then M admits $n = \dim M$ mutually orthogonal homogeneous geodesics through the origin $o = eK$.*

Homogeneous spaces all of whose geodesics are such orbits have special importance:

Definition. A *g.o. space* is a Riemannian manifold (M, g) all of whose geodesics are orbits of one-parameter subgroups of isometries. This means that there exists a transitive group G of isometries such that $M = G/K$, and so that every geodesic in M is of the form $(\exp tX) \cdot p$, $(X \in \mathfrak{g}$, $p \in M)$.

Note that we can always choose G to be $I_0(M)$. Also, it is enough to check whether all geodesics through a single point (say $o = eK$) are of the form $\gamma(t) = (\exp tX) \cdot o$. Indeed, any other point in M is of the form $a \cdot o$, with $a \in G$, and the geodesics through $a \cdot o$ are then of the form $a\gamma(t) = \exp(t\mathrm{Ad}(a)X) \quad (a \cdot o)$.

Definition. A nonzero element X in $\mathfrak{g}$ is called a *geodesic vector* if the curve $\gamma(t) = (\exp tX) \cdot o$ is a geodesic.

The following proposition gives a characterization of homogeneous geodesics in terms of geodesic vectors.

Proposition 8.10 ([**Kos**], [**Vi**], [**Kow-Va**]). *Let $M = G/K$ be a Riemannian homogeneous space. Then the orbit $\gamma(t) = (\exp tX) \cdot o$ is a geodesic in M if and only if one of the following conditions is fulfilled:*

(a) $[X, A(X_{\mathfrak{m}})] \in \mathfrak{k}$.
(b) $\langle [X_{\mathfrak{k}}, X_{\mathfrak{m}}], Y\rangle = \langle X_{\mathfrak{m}}, [X_{\mathfrak{m}}, Y]_{\mathfrak{m}}\rangle$ *for all* $Y \in \mathfrak{m}$.
(c) $\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0$ *for all* $Y \in \mathfrak{m}$.

As usual $X_{\mathfrak{m}}$ and $X_{\mathfrak{k}}$ denote the components of $X \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ in $\mathfrak{m}$ and $\mathfrak{k}$ respectively.

Low-dimensional examples.

In an attempt to clarify the question of scarcity of g.o. spaces in comparison with the naturally reductive spaces, O. Kowalski and L. Vanhecke in [**Kow-Va, Theorems 4.1, 4.4, 5.3**], classified all Riemannian homogeneous spaces of dimension ≤ 6. In higher dimensions the problem is generally open. Their results can be summarized as follows:

Theorem 8.11. *Let $M = G/K$ be a Riemannian homogeneous g.o. space of dimension n.*

(1) *If* $n \leq 4$, *then* M *is naturally reductive.*

(2) *If* $n = 5$, *then* M *is either naturally reductive or of "isotropy" type* $SU(2)$. *These are certain homogeneous spaces of the form* $G/SU(2)$, *either compact or non-compact. Furthermore, they are either naturally reductive, or it is possible to express* M *in the form* $G'/U(2)$, *so that they become naturally reductive.*

(3) *If* $n = 6$, *it is possible to give a list of all simply connected such spaces that are in no way naturally reductive (that is, in any group extension of* G*). In the compact case, there is only the homogeneous space* $SU(5)/U(2)$.

The example $SU(5)/U(2)$ is the first example of a compact simply connected Riemannian space which is in no way naturally reductive. The non-compact examples for the case $n = 6$ are closely related to Kaplan's 6-dimensional example. For other recent results on g.o. spaces we refer to [**Alek-Ar**], [**Ma**] and [**Dus-Kow-Ni**].

Bibliography

[**Ad**] J. F. Adams: *Lectures on Lie Groups*, Benjamin, New York, 1966.

[**Alek1**] D. V Alekseevsky: *Flag Manifolds*, in: Sbornik Radova, Vol. 6, Beograd, (1997) 3–35.

[**Alek2**] D. V. Alekseevsky: *Homogeneous Einstein metrics*, in: Differential Geometry and its Applications (Proceedings of Brno Conference), Univ. of J. E. Purkyne, Czechoslovakia (1987), 1–21.

[**Alek-Ar**] D. V. Alekseevsky and A. Arvanitoyeorgos: *Metrics with homogeneous geodesics on flag manifolds*, to appear in Comment. Math. Univ. Carolinae 43 (2) (2002), 189–199.

[**Alek-Pe**] D. V. Alekseevsky and A. M. Perelomov: *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl. 20 (3) (1986), 171–182.

[**Am-Si**] W. Ambrose and I. M. Singer: *On homogeneous Riemannian manifolds*, Duke Math. J. 20 (1958), 647–669.

[**Ar1**] A. Arvanitoyeorgos: *New invariant Einstein metrics on generalized flag manifolds*, Trans. Amer. Math. Soc. 337 (2) (1993), 981–995.

[**Ar2**] A. Arvanitoyeorgos: *Homogeneous Einstein metrics on Stiefel manifolds*, Comment. Math. Univ. Carolinae 37 (3) (1996), 627–634; Erratum: 37 (4) (1996).

[**Arn**] V. Arnold: *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978.

[**At1**] M.F. Atiyah: *Convexity and commuting Hamiltonians*, Bull. Lond. Math. Soc. 16 (1982), 1–15.

[**At2**] M. F. Atiyah (editor): *Representation Theory of Lie Groups*, Cambridge University Press, London Math. Soc. Lecture Notes 34, 1979.

[**At-Bo**] M. Atiyah and R. Bott: *The moment map and equivariant cohomology*, Topology 23 (1) (1984), 1–28.

[**B**] A. Borel: *Kählerian coset spaces of semisimple Lie groups*, Proc. Nat. Acad. Sci. USA, 40 (1954), 1147–1151.

[**Bo-Di**] W. E. Boyce and R. C. DiPrima: *Elementary Differential Equations*, John Wiley, New York (1992).

[**B-F-R**] M. Bordeman and M. Forger and H. Römer: *Homogeneous Kähler manifolds: paving the way towards new supersymmetric sigma models*, Comm. Math. Phys. 102 (1986), 605-647.

[**Be**] A. Besse: *Einstein Manifolds*, Springer-Verlag, Berlin 1986.

[**Ber1**] M. Berger: *Encounter with a Geometer, Part II*, Notices Amer. Math. Soc. 47 (3) (2000), 326–340.

[**Ber2**] M. Berger: *Sur les variétés d' Einstein compactes*, Comptes rendus de la III^{e} Réunion du groupement des mathématiciens d' expression latine, Louvain, Belgique (1966), 35–55.

[**Ber3**] M. Berger: *Quelques formules de variation pour une structure riemannienne*, Ann. Scien. Éc. Norm. Sup. 3 (4) (1970), 285–294.

[**Bo**] R. Bott: *An application of Morse theory to the topology of Lie groups*, Bull. Soc. Math. France 84 (1956), 251–281.

[**Bo-Sa**] R. Bott and H. Samelson: *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. 80 (1958), 964–1029.

[**B-H**] A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces I, II, III*, Amer. J. Math. 80 (1958), 458–538; 81 (1959), 315–382; 82 (1960), 491–504.

[**Br-Cl**] F. Brickell and R. S. Clark: *Differentiable Manifolds*, Van Nostrand Reinhold Company, New York, 1970.

[**Brö-TD**] T. Bröcker and T. Tom Dieck: *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985.

[**C-Ch-La**] S.S. Chern and W. H. Chen and K. S. Lam: *Lectures on Differential Geometry*, World Scientific, Singapore (2000).

[**Ca-Se-Mc**] R. Carter and G. Segal and I. Macdonald: *Lectures on Lie groups and Lie Algebras*, London. Math. Soc. Student Texts 32, Cambridge University Press, 1995.

[**Ch-Eb**] J. Cheeger and D. G. Ebin: *Comparison Theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[**Ch**] C. Chevalley: *Theory of Lie Groups*, Princeton Univ. Press, Princeton, New Jersey, 1946.

[**Cu**] M. L. Curtis: *Matrix Groups*, Springer-Verlag, New York, 1979.

[**DC**] M. P. Do Carmo: *Riemannian Geometry*, Birkhäuser, Boston, 1992.

[**Du-Ko**] J. J. Duistermaat and J. A. C. Kolk: *Lie Groups*, Springer-Verlag, Berlin, 2000.

[**Dus1**] Z. Dušek: *Structure of geodesics in a* 13*-dimensional group of Heisenberg type*, Proc. Coll. Differential Geom. in Debrecen (2001), 95–103.

[**Dus2**] Z. Dušek: *Explicit geodesic graphs on some H-type groups*, preprint.

[**Dus-Kow-Ni**] Z. Dušek and O. Kowalski and S. Ž. Nikčević: *New examples of g.o. spaces in dimension 7* (preprint).

[**Fe**] H. D. Fegan: *Introduction to Compact Lie Groups*, Word Scientific, Singapore, 1991.

[**Fr-Uh**] D. S. Freed and K. K. Uhlenbeck (editors): *Geometry and Quantum Field Theory*, Amer. Math. Soc. IAS/Park City Mathematics Series Vol. 1 (1995).

[**Fra**] T. Frankel: *Fixed points and torsion on Kähler manifolds*, Annals of Math. 70 (1959), 1–8.

[**Frö**] A. Fröhlicher: *Zur Differential geometrie der komplexen structuren*, Math. Ann. 129 (1955), 50–95.

[**Fu-Ha**] W. Fulton and J. Harris: *Representation Theory of Compact Lie Groups*, A first course, Springer-Verlag, New York, 1991.

[**Ga-Hu-La**] S. Gallot and D. Hulin and J. Lafontaine: *Riemannian Geometry*, Springer-Verlag, New York, 1987.

[**Go**] C. S. Gordon: *Homogeneous manifolds whose geodesics are orbits*, in: Topics in Geometry, in Memory of Joseph D'Atri, Birkhäuser, Basel, (1966), 155–174.

[**Gu1**] M. A. Guest: *Harmonic Maps, Loop Groups and Integrable Systems*, London Math. Soc. Student Texts 38, Cambridge University Press, 1997.

[**Gu2**] M. A. Guest: *Geometry of maps between generalized flag manifolds*, J. Differential Geom. 25 (1987), 223–247.

[**He**] S. Helgason: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Amer. Math. Soc., 2001.

[**Hi**] D. Hilbert: *Die Grundlagen der Physik*, Nachr. Ges. Wiss. Gött. (1915), 395–405.

[**Hs**] W. Y. Hsiang: *Lectures on Lie Groups*, World Scientific, Singapore, 2000.

[**Hu**] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[**Ja**] I. M. James: *The Topology of Stiefel Manifolds*, Cambridge Univ. Press, Lecture Notes Series 24, Great Britain, 1976.

[**Jac**] N. Jacobson: *Exceptional Lie Algebras*, Marcel Dekker, New York, 1971.

[**Je**] G. R. Jensen: *Einstein metrics on principal fibre bundles*, J. Differential Geom. 8 (1973), 599–614.

[**Jos**] J. Jost: *Riemannian Geometry and Geometric Analysis*, Springer-Verlag, New York, 1995.

[**Ka**] A. Kaplan: *On the geometry of groups of Heisenberg type*, Bull London Math. Soc. 15 (1983), 35–42.

[**Kaj**] V V. Kajzer: *Conjugate points of left-invariant metrics on Lie groups*, Sov. Math. 34 (1990), 32–44.

[**Kaw**] K. Kawakubo: *The Theory of Transformation Groups*, Oxford University Press, 1991.

[**Ke1**] M. M. Kerr: *Some new homogeneous Einstein metrics on symmetric spaces*, Trans. Amer. Math. Soc. 348 (1) (1996), 153–171.

[**Ke2**] M. M. Kerr: *New examples of homogeneous Einstein metrics*, Michigan Math. J. 45 (1998), 115–134.

[**Ki**] M. Kimura: *Homogeneous Einstein metrics on certain Kähler C-spaces*, Adv. Stud. Pure Math., 18-I (1990), 303–320.

[**Kn**] A. Knapp: *Lie Groups, Lie Algebras and their Cohomology*, Princeton University Press, 1988.

[**Ko-No**] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry*, Wiley (Interscience), New York, Vol. I, 1963; Vol. II, 1969.

[**Kos**] B. Kostant: *Holonomy and Lie algebra of motions in Riemannian manifolds*, Trans. Amer. Math. Soc. 80 (1955), 520–542.

[**Kow-Ni**] O. Kowalski and S. Ž. Nikęvić: *On geodesic graphs of Riemannian g.o.spaces*, Arch. Math. 73 (1999), 223–234.

[**Koz1**] J. L. Koszul: *Sur la form Hermitienne canonique des espaces homogènes complexes*, Canad. J. Math. 7 (1955), 562–576.

[**Koz2**] J. L. Koszul: *Exposés sur les espaces homogènes symétriques*, Soc. Mat. São Paolo (1959), 1–71.

[**Kow-Ni-Vl**] O. Kowalski and S. Ž. Nikęvić and Z. Vlášek: *Homogeneous geodesics in homogeneous Riemannian manifolds – examples*, in: Geometry and Topology of Submanifolds, X (Beijing-Berlin, 1999), World Scientific, New York, (2000), 104–112.

[**Kow-Sz**] O. Kowalski and J. Szenthe: *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Geom. Ded. 81 (2000), 209–214; Erratum: 84 (2001), 331–332.

[**Kow-Va**] O. Kowalski and L. Vanhecke: *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Bollettino Un. Math. Ital. B (7) 5 (1991), 189–246.

[**Ku-Ri-Ru**] Yu. A. Kubyshin and O. Richter and G. Rudolph: *Invariant connections on homogeneous spaces*, J. Math. Phys. 34 (11) (1993), 5268–5282.

[**Kü**] W. Kühnel: *Differential Geometry: Curves-Surfaces-Manifolds*, Amer. Math. Soc. Student Math. Lib. 16, 2002

[**Ku**] M. Kuga: *Galois' Dream: Group Theory and Differential Equations*, S. A. Books Inc., Portland, 1994.

[**L-Wa**] C. LeBrun and M. Wang (editors): *Surveys in Differential Geometry Volume VI: Essays on Einstein Manifolds*, International Press, 1999.

[**Lo**] O. Loos: *Symmetric Spaces*, Vols I-II, Benjamin, New York, 1969.

[**Ma**] R. A. Marinosci: *Homogeneous geodesics in a three-dimensional Lie group*, Comment. Math. Univ. Carolinae, 43 (2) (2002), 261–270.

[**Mi1**] J. Milnor: *Differentiable structures on spheres*, Amer. J. Math. 81 (1959), 962–972.

[**Mi2**] J. Milnor: *Curvature of left invariant metrics on Lie groups*, Adv. in Math. 21 (1976), 293–329.

[**Mi-St**] J. Milnor and J. Stasheff: *Characteristic Classes*, Annals of Mathematical Studies, no. 76, Princeton University Press, Princeton, New Jersey, 1974.

[**Mu**] Y. Mutō: *On Einstein metrics*, J. Differential Geom. 9 (1974), 521–530.

[**Na**] M. Nakahara: *Geometry, Topology, and Physics*, Institute of Optics, Bristol-Philadelphia, 1990.

[**Nis**] M. Nishiyama: *Classification of invariant complex structures on irreducible compact simply connected coset spaces*, Osaka J. Math. 21 (1984), 39–58.

[**Ok**] T. Okubo: *Differential Geometry*, Marcel Dekker, 1987.

[**ON**] B. O'Neill: *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, United Kingdom, 1983.

[**Oni**] A. L. Oniščik: *Transitive compact transformation groups*, Math. Sb. 60 (102) (1963), 447–485, (English translation: Amer. Math. Soc. Transl. 55 (2) (1966), 153–194).

[**Pa**] R. Palais: *The principle of symmetric criticality*, Comm. Math. Physics 69 (1979), 19–30.

[**Pe**] A. M. Perelomov: *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser, 1990.

[**Pr-Se**] A. N. Pressley and G. Segal: *Loop Groups*, Oxford Univ. Press, London, 1987.

[**Ra**] S. Ramanujam: *Application of Morse theory to some homogeneous spaces*, Tôhoku Math. J. 21 (1969), 343–353.

[**Ri**] C. Riehm: *The automorphism group of a composition of quadratic forms*, Trans. Amer. Math. Soc. 269 (1982), 403–414.

[**Ro1**] E. D. Rodionov: *Structure of standard homogeneous Einstein manifolds with simple isotropy group.I*, Siberian Math. J. 37 (1) (1996), 151–167.

[**Ro2**] E. D. Rodionov: *Structure of standard homogeneous Einstein manifolds with simple isotropy group.II*, Siberian Math. J. 37 (3) (1966), 542–551.

[**Sa**] H. Samelson: *Notes on Lie Algebras*, Springer-Verlag, New York, 1990.

[**Sa-Wa**] A. A. Sagle and R. E. Walde: *Introduction to Lie Groups and Lie Algebras*, Academic Press, United Kingdom, 1973.

[**Sag**] A. A. Sagle: *Some homogeneous Einstein manifolds*, Nagoya Math. J. 39 (1970), 81–106.

[**Sak**] Y. Sakane: *Homogeneous Einstein metrics on flag manifolds*, Lobachevskii J. Math. 4 (1999), 71–87.

[**Se**] J-P. Serre: *Complex Semisimple Lie Algebras*, Springer-Verlag, New York, 1987.

[**Spi**] M. Spivak: *A Comprehensive Introduction to Differential Geometry*, Vols. I-V, Publish or Perish, Berkley, California, 1970, 1975.

[**Sh**] R. W. Sharpe: *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer-Verlag, New York, 1997.

[**Si**] B. Simon: *Representations of Finite and Compact Groups*, Amer. Math. Soc. Graduate Studies in Mathematics 10, 1996.

[**Sie**] J. Siebenthal: *Sur certains modules dans une algèbre de Lie semisimple*, Comment. Math. Helv. 44 (1) (1964), 1–44.

[**Sz**] J. Szenthe: *Homogeneous geodesics of left-invariant metrics*, Univers. Iagellonicae Acta Mathematica, Fasciculus XXXVIII (2000), 99–103.

[**Va**] S. Varadarayan: *Lie Groups, Lie Algebras and Their Representations*, Springer-Verlag, New York, 1984.

[**Vi**] E. B. Vinberg: *Invariant linear connections in a homogeneous manifold*, Trudy MMO 9 (1960), 191–210.

[**Wa1**] N.R. Wallach: *Compact homogeneous Riemannian manifolds with strictly positive sectional curvature*, Ann. of Math. 96 (1972), 277–295.

[**Wa2**] N. R. Wallach: *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1972.

[**Wan**] Z. Wan: *Lie Algebras*, Pergamon Press, Oxford, 1975.

[**Wa-Zi1**] M. Wang and W. Ziller: *On normal homogeneous Einstein metrics*, Ann. Sci. Ec. Norm. Sup. 18 (1985), 563–633.

[**Wa-Zi2**] M. Wang and W. Ziller: *Existence and non-existence of homogeneous Einstein metrics*, Inventiones Math. 84 (1986), 177–194.

[**Wg**] H. C. Wang: *Closed manifolds with homogeneous complex structures*, Amer. J. Math. 76 (1954), 1–32.

[**War**] F. Warner: *Foundations of Differential Manifolds and Lie Groups*, Springer-Verlag, New York, 1983.

[**Wo1**] J. A. Wolf: *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. 120 (1968), 59–148; Correction: 152 (1984), 141–142.

[**Wo2**] J. A. Wolf: *Spaces of Constant Curvature*, McGraw-Hill, New York, 1962.

[**Wo-Gr**] J. A. Wolf and A. Gray: *Homogeneous spaces defined by Lie group automorphisms II*, J. Differential Geom. 2 (1968), 115–159.

[**Wi**] E. Witten: *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), 661.

[**Wil**] T. J. Willmore: *Riemannian Geometry*, Clarendon Press, Oxford, 1993.

[**Zi**] W. Ziller: *Homogeneneous Einstein metrics*, in: Global Riemannian Geometry, T. J. Willmore - N. J. Hitchin Eds, John Wiley (1984), 126–135.

[**Zu**] L. Zulli: *Charting the* 3*-Sphere – An Exposition for Undergraduates*, Amer. Math. Monthly, March 1996, 221–229.

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It is remarkable that so much about Lie groups could be packed into this small book. But after reading it, students will be well-prepared to continue with more advanced, graduate-level topics in differential geometry or the theory of Lie groups.

The theory of Lie groups involves many areas of mathematics. In this book, Arvanitoyeorgos outlines enough of the prerequisites to get the reader started. He then chooses a path through this rich and diverse theory that aims for an understanding of the geometry of Lie groups and homogeneous spaces. In this way, he avoids the extra detail needed for a thorough discussion of other topics.

Lie groups and homogeneous spaces are especially useful to study in geometry, as they provide excellent examples where quantities (such as curvature) are easier to compute. A good understanding of them provides lasting intuition, especially in differential geometry.

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