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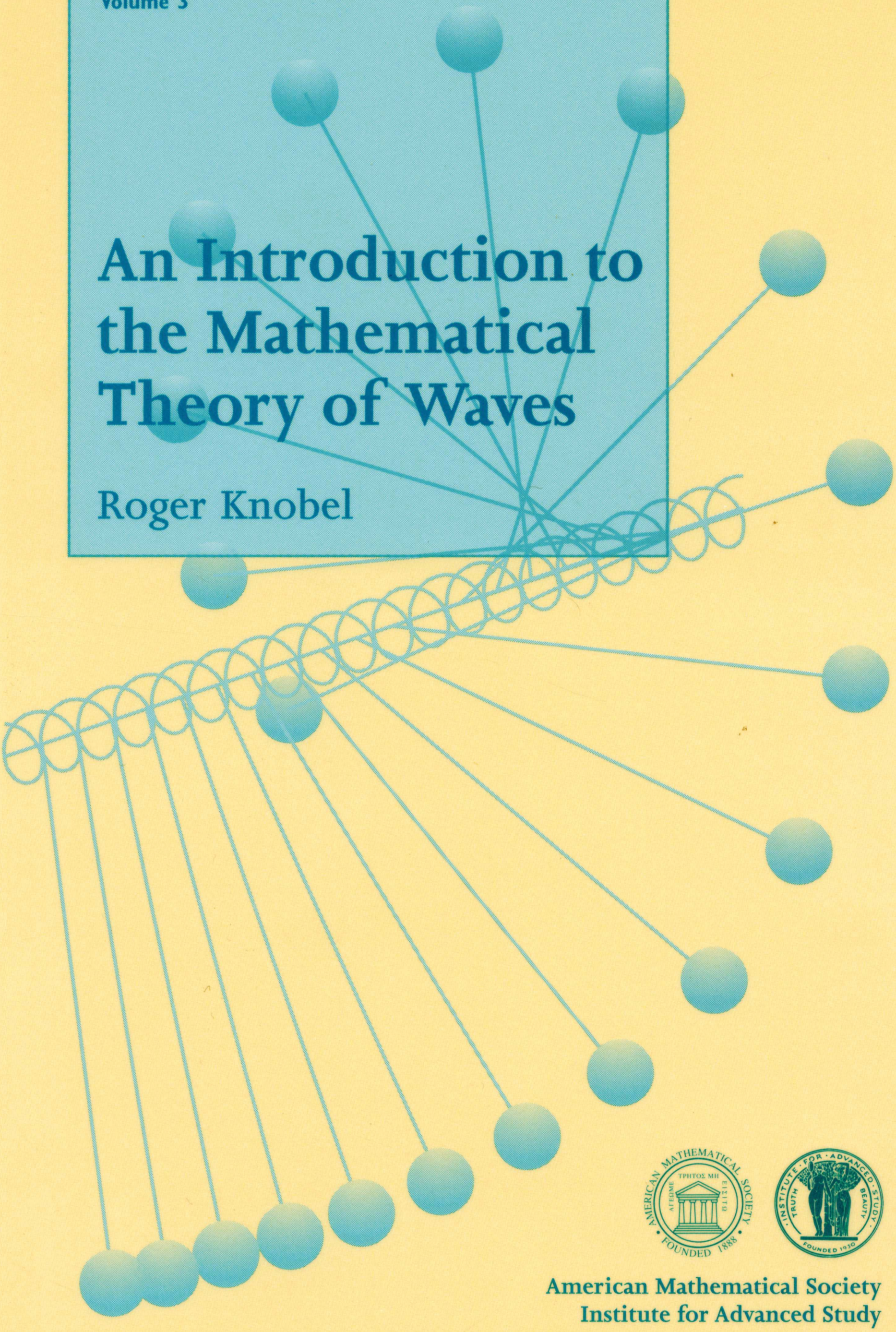
An Introduction to the Mathematical Theory of Waves



Roger Knobel

American Mathematical Society
Institute for Advanced Study

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An Introduction to the Mathematical Theory of Waves

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An Introduction to the Mathematical Theory of Waves

Roger Knobel

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2000 *Mathematics Subject Classification.* Primary 35–01; Secondary 00–01, 73–01.

ABSTRACT. This book provides an introduction to basic terminology and concepts found in mathematical studies of wave phenomena. Wave forms such as traveling waves, solitons, standing waves, and shock waves are studied by constructing solutions of partial differential equations. Particular examples of wave behavior are given through fundamental equations, including the KdV equation, Klein-Gordon equation, wave equation, Burgers' equation, and several traffic flow models.

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IAS/Park City Mathematics Institute

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid 1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey. The PCMI will continue to hold summer programs alternately in Park City and in Princeton.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, mathematics education researchers, and undergraduate faculty. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the High School Teacher Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Program deal with this topic as well. Lecture notes from the Graduate Summer School are published each year in the IAS/Park City Mathematics Series. Course materials from the Undergraduate Program, such as the current volume, are now being published as part of the IAS/Park City Mathematical Subseries in the Student Mathematical Library. We are happy to make available more of the excellent resources which have been developed as part of the PCMI.

At the summer institute late afternoons are devoted to seminars of common interest to all participants. Many deal with current issues in education; others treat mathematical topics at a level which encourages broad participation. The PCMI has also spawned interactions between universities and high schools at a local level. We hope to share these activities with a wider audience in future volumes.

Robert Bryant and Dan Freed, Series Editors

May, 1999

Preface

Background. These notes are based on an undergraduate course given in July 1995 at the Park City Mathematics Institute's summer program on Nonlinear Waves. The undergraduate course on Linear and Nonlinear Waves consisted of a series of lectures, problem sessions, and computer labs centered around the mathematical modeling and analysis of wave phenomena. This book was written to reflect the content and nature of the course, intermixing discussion with exercises and computer experiments.

The intent of this book is to provide an introduction to basic terminology and concepts found in mathematical studies of wave phenomena. The level of this material is aimed at someone who has completed a basic calculus sequence through multi-variable calculus, and preferably completed a beginning course in ordinary differential equations. Concepts from partial differential equations are introduced as needed and no prior experience with this topic is assumed.

Companion Software. Several problems within these notes are best analyzed with mathematical software to perform plots and visualizations of functions of two variables. Most mathematical software packages are sufficient, although selected exercises refer to particular files which are to be used with MATLAB®. These supplemental files were originally written for use at the PCMI summer program and have been incorporated here as exercises. Readers who use MATLAB[1] are encouraged to obtain these supplemental files through The MathWorks anonymous FTP site at the address

ftp://ftp.mathworks.com/pub/books/knobel.

Further Reading. The following books were influential in the writing of these notes and are recommended for readers wishing to supplement or expand the material presented here. The text *Partial Differential Equations for Engineers and Scientists* by Stanley Farlow provides a basic introduction to partial differential equations. Much of David Logan's book *An Introduction to Nonlinear Partial Differential Equations* pertains to the wave behavior arising from nonlinear partial differential equations discussed here. A full collection of material and exercises for the models of traffic flow discussed later in these notes can be found in the textbook *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow* by Richard Haberman. His text *Elementary Applied Partial Differential Equations* expands upon much of the material of these notes, with many sections devoted to various aspects of wave theory. Finally, a more advanced reference on the mathematical theory of linear and nonlinear wave phenomena can be found in the book *Linear and Nonlinear Waves* by G.B. Whitham. See the bibliography for these and other suggested sources of further reading.

Acknowledgments. I am indebted to those who gave me the opportunity, support, and assistance to complete these notes. In particular I would like to extend my appreciation to the other undergraduate instructors at the 1995 IAS/PCMI summer program, Steve Cox and Richard Palais, for their advice and collaboration in the design of the PCMI course; the IAS/PCMI for giving me the opportunity to participate in the summer program; and finally to Monty B. Taylor and my wife Mayra for their assistance in reviewing the manuscript.

Roger Knobel

[1]For MATLAB product information, please contact: The MathWorks, Inc., 3 Apple Hill Drive, Natick, MA 01760, USA, Tel. 508-647-7000, Fax 508-647-7101, E-mail info@mathworks.com, Web www.mathworks.com.

Part 1

Introduction

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Chapter 1

Introduction to Waves

At this time we should discuss how we hear. The same thing happens in sounds that happens when a stone, thrown from above, falls into a puddle or into quiet water. First it causes a wave in a very small circle; then it disperses clusters of waves into larger circles, and so on until the motion, exhausted by the spreading out of waves, dies away. The latter, wider wave is always diffused by a weaker impulse. Now if something should impede the spreading waves, the same motion rebounds immediately, and it makes new circles by the same undulations as at the center whence it originated.

In the same way, then, when air that is struck creates a sound, it affects other air nearby and in this way sets in motion a circular wave of air; and so it is diffused and reaches the hearing of all standing around at the same time. The sound is fainter to someone standing at a distance, since the wave of activated air approaches him more weakly.[1]

Boethius, *De institutione musica*, 500 A.D.

[1] From *Fundamentals of Music* by Anicius Manlius Severinus Boethius, translated by Calvin M. Bower, edited by Claude Palisca, Yale University Press, 1989, p. 21.

Debate continues today on whether Boethius was executed in 524 A.D. because of or in spite of his being too smart.

1.1. Wave phenomena

The notion of a wave is something familiar to everyone in one form or another, whether it be ocean waves, sound waves, a wave good-bye, or the "wave" at a football game. The broad use of the term wave, however, makes it difficult to produce a precise definition of a wave. Instead of attempting to state a single mathematical definition, we will be guided by an intuitive point of view to identify and describe wave phenomena.

In many cases an observed wave is the result of a disturbance moving through a medium such as water, air, or a crowd of people. As the disturbance is transferred from one part of the medium to another, we are able to observe the location of the disturbance as it moves with speed in a particular direction. Any quantitative measurement or feature of the medium which clearly identifies the location and velocity of the disturbance is called a **signal**. The signal may distort; however, as long as it remains recognizable, it can be used to identify the motion of the disturbance. It is on these ideas that we will base our intuitive notion of a wave:

> A **wave** is any recognizable signal that is transferred from one part of a medium to another with a recognizable velocity of propagation. [**Whi**, p. 2]

The terms *medium*, *signal*, and *transferred* are used in a sufficiently broad manner to allow a wide range of interpretations of what constitutes a wave.

1.2. Examples of waves

Disturbances spreading through a medium are an important phenomenon in diverse fields such as acoustics, biology, chemistry, electromagnetics, mechanics, and fluid mechanics. In observing wave phenomena, one should always be able to identify a feature (signal) of a disturbance which is transferred from one part of the medium to another with a recognizable velocity.

Figure 1.1. Ripples on a pond. Extreme water height signals the location and movement of a wave.

One example commonly used to illustrate wave phenomena is the rings radiating from a point on the surface of water, such as those formed by throwing a stone into a still pond (see Figure 1.1). The stone creates an initial disturbance in a small region of the pond's surface. Crests and troughs form as the disturbance is transferred horizontally through this layer of water, creating rings which appear to radiate outward from the initial disturbance. An identifiable feature of each ring is its crest, which is an extreme vertical displacement of water. This feature is a signal which moves outward from the center with a recognizable velocity of propagation, identifying a wave. Note that while the wave moves horizontally across the pond, the signal itself is formed by the vertical displacement of water.

Another phenomena which is also classified as a wave is that of traffic backing up at a stop light (see Figure 1.2). Viewed from above, the end of a line of stopped traffic appears to move away from the light with a recognizable velocity as approaching automobiles reach the end of the line. In this case the medium is automobile traffic and the disturbance is stopped traffic. One recognizable signal of this disturbance is the abrupt increase in traffic density (number of cars per mile) that drivers sense when they reach the back of the line. Note that while the individual cars are moving towards the stop light, the signal (the end of the line) propagates away from the light.

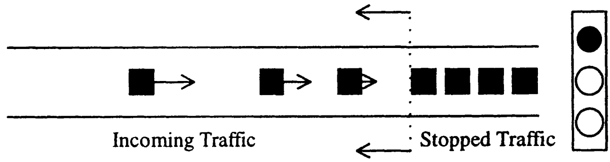


Figure 1.2. The end of a line of stopped traffic appears to move backwards through incoming traffic.

Table 1.1. Common Wave Phenomena

Wave Phenomena	Medium	Signal
Ripples on a pond	Surface layer of water	Extreme water surface height
Compression waves	Elastic bar	Maximum longitudinal displacement
Sound waves	Gas or liquid	Pressure extrema
Shock waves	Gas	Abrupt change in pressure
Traffic waves	Car traffic on a road	Abrupt change in traffic density
Epizootic waves (geographic spread of disease)	A population susceptible to contracting a disease	Extreme number of infectives

Additional examples of wave phenomena and their signals are shown in Table 1.1. See also [**BB**, p. 16] for a more extensive list and discussion of different examples of waves.

Chapter 2

A Mathematical Representation of Waves

The purpose of this chapter is to illustrate the use of functions of two variables as a way of representing and visualizing waves moving in a one-dimensional medium.

2.1. Representation of one-dimensional waves

Our study will be of waves propagating in one-dimensional media such as strings, long thin pipes, and single-lane roads. The medium itself might move or distort in two or three dimensions; however, the location of a wave's signal can be described by a single coordinate along a line.

One-dimensional waves are represented mathematically by functions of two variables $u(x,t)$, where u represents the value of some quantitative measurement made at every position x in the medium at time t. At a fixed time t_0, viewing $u(x,t_0)$ as a function of x can indicate the existence of a disturbance in the medium. As shown in Figure 2.1, signals which most easily identify the presence and location of a disturbance are extreme points or abrupt changes in the value of $u(x,t_0)$. Examining $u(x,t_k)$ at later times $t_1 < t_2 < \ldots$ indicates how the signal is moving through the medium (Figure 2.2).

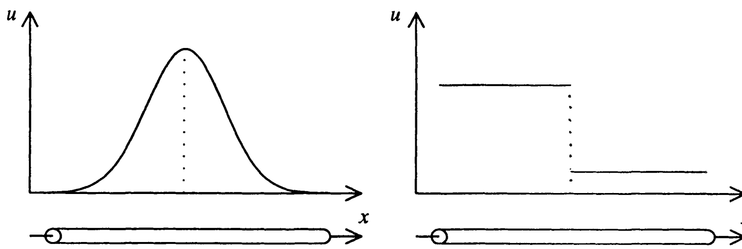


Figure 2.1. Two signals which identify the location of a disturbance in a medium: an extreme point and an abrupt change in the value of $u(x, t_0)$.

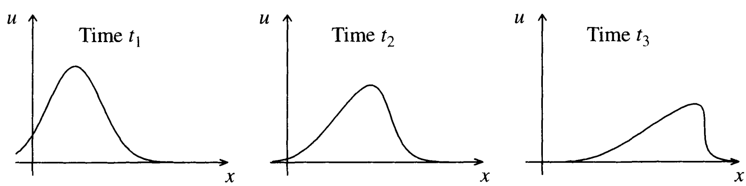
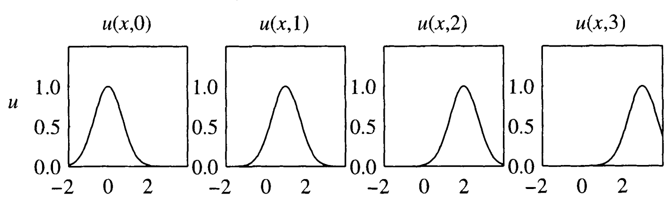


Figure 2.2. A graphical representation of a disturbance moving through a medium.

Exercise 2.1. Suppose that at time t, the value of u at position x in a one-dimensional medium is given by $u(x,t) = e^{-(x-t)^2}$. Snapshots of the wave represented by this function at times $t = 0, 1, 2, 3$ are



and illustrate a wave moving right with constant speed. Modify the function $u(x, t)$ so that

(a) The wave moves to the left.

(b) The wave moves to the right with increasing speed.

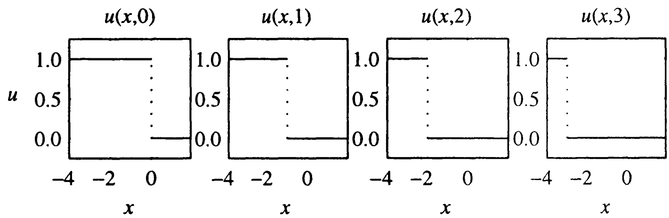
(c) The amplitude (height) of the wave decreases as the wave moves to the right.

Exercise 2.2. The Heaviside or unit step function $H(x)$ is defined to be

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

(a) Sketch the graph of $u(x,t) = H(x-t)$ in the xu-plane at times $t = 0, 1, 2, 3$.

(b) Use the Heaviside function to construct a function $u(x,t)$ which could be used to model the wave



2.2. Methods for visualizing functions of two variables

There are several ways in which graphs can be used to give a visual representation of a function of two variables. These visualizations often make it easier to identify the location and movement of a disturbance within a medium. In this section we will describe four visualization methods: animation, slice plots, surface plots, and xt-diagrams.

Animation. At any fixed time t_0, $u(x, t_0)$ is a function of one variable x whose graph in the xu-plane is a *profile* of u along the medium at time t_0. By plotting the graph of $u(x, t_k)$ in the xu-plane over a sequence of times $t_1 < t_2 < \ldots$, one can construct a sequence of *frames.* Placing these frames side-by-side or playing them quickly on a computer creates the effect of motion (Figure 2.3).

Slice Plots. Viewing several frames of animation at once helps view the history of the disturbance over time. By placing frames of animation in a three-dimensional coordinate system with x, t, and u axes, one can look at several frames in a **slice plot** (Figure 2.4).

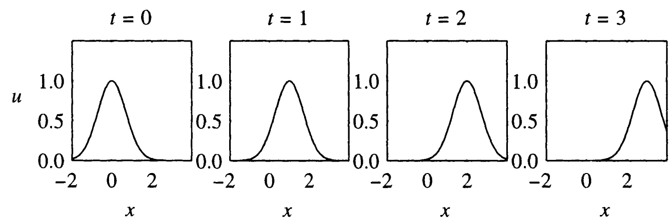


Figure 2.3. Animation of $u(x,t) = \exp(-(x-t)^2)$.

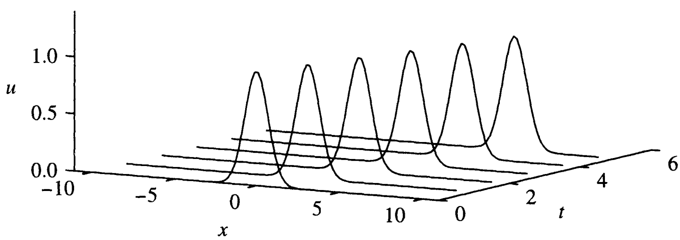


Figure 2.4. Slice plot view of $u(x,t) = \exp(-(x-t)^2)$.

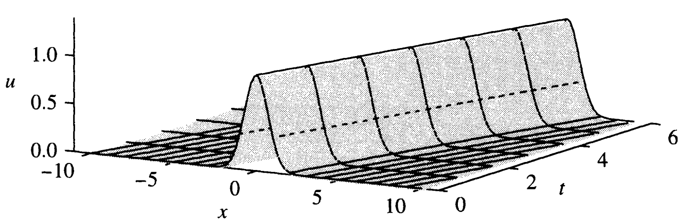


Figure 2.5. Surface plot of $u(x,t) = \exp(-(x-t)^2)$.

Surface Plots. The function $u(x,t)$ can be plotted as a surface of points $(x,t,u(x,t))$ in the three-dimensional xtu coordinate system. This is the same as taking a slice plot and letting t vary continuously over an interval of values (Figure 2.5).

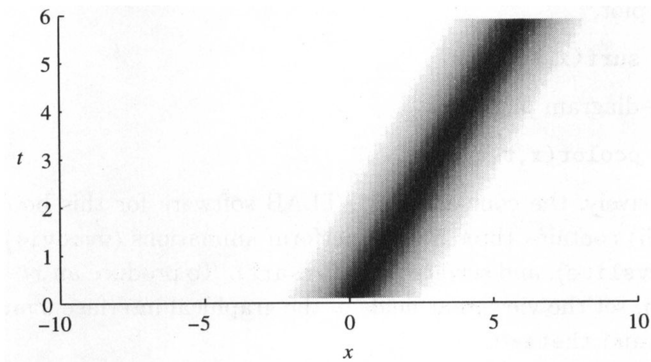


Figure 2.6. xt–plane view of $u(x,t) = \exp(-(x-t)^2)$.

xt–Diagram. Looking down onto the xt–plane, the value of $u(x,t)$ at each point (x,t) can be described by shades of grey or color to indicate the magnitude of u (Figure 2.6). This is called an **xt–diagram** or **density plot** of $u(x,t)$ and is similar to viewing a surface plot of $u(x,t)$ from above.

Several mathematical software packages can produce each of the four types of plots. In MATLAB, for example, the functions `movie`, `waterfall`, `surf`, and `pcolor` can be used to produce animation, slice, surface, and xt–diagram plots. First compute u on a mesh of points (x,t),

```
x = -10:0.1:10; t = 0:0.3:6;
[X,T] = meshgrid(x,t);
u = exp(-(X-T).^2);
```

The function u can then be visualized by animation,

```
M = moviein(length(t));
for j=1:length(t),
  plot(x,u(j,:)), M(:,j)=getframe;
end;
movie(M)
```

slice plot,

```
waterfall(x,t,u)
```

surface plot,

```
surf(x,t,u)
```

and xt–diagram plot,

```
pcolor(x,t,u).
```

Alternatively, the companion MATLAB software for this book (see page xiii) contains three files to perform animations (**wvmovie**), slice plots (**wvslice**), and surface plots (**wvsurf**). To produce an xt–plane diagram, set the viewpoint fields in the graphical interface **wvsurf** to **phi=90** and **theta=0**.

In Maple V®, the four types of plots can be constructed by first defining $u(x,t)$ and then loading the plots package at the Maple prompt > by

```
> u := (x,t) -> exp(-(x-t)^2);
> with(plots):
```

The function u can then be visualized by animation,

```
> animate(u(x,t), x=-10..10, t=0..6);
```

slice plot,

```
> slices := {seq([x,t,u(x,t)], t={0,2,4,6})}:
> spacecurve(slices, x=-10..10);
```

surface plot,

```
> plot3d(u(x,t), x=-10..10, t=0..6);
```

or xt–diagram

```
> densityplot(u(x,t), x=-10..10, t=0..6);
```

Exercise 2.3. Using computer software, reproduce the animation represented by Figure 2.3 and the plots shown in Figures 2.4–2.6 for the function $u(x,t) = e^{-(x-t)^2} = \exp(-(x-t)^2)$.

Exercise 2.4. View the function $u(x,t) = e^{-(x-t)^2} + e^{-(x+t)^2}$ by animation, slice plots, surface plots, and an xt–diagram.

Chapter 3

Partial Differential Equations

3.1. Introduction and examples

Our mathematical description of waves will be through functions of two variables $u(x,t)$. If we think of u as being the value of some measurement of a one-dimensional medium at position x and time t, then partial derivatives of u with respect to x and t have important physical meanings as rates of change. Scientific principles or observations about how u changes can often be expressed as an equation which relates u and its derivatives.

A **partial differential equation** (PDE) for a function $u(x,t)$ is a differential equation that involves one or more of the partial derivatives of u with respect to x and t. We will usually denote the partial derivatives of u by

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_{xt} = \frac{\partial^2 u}{\partial t \partial x}, \quad \ldots$$

Partial differential equations are a fundamental mathematical tool for modeling physical phenomena which change over time. The partial differential equations in the following examples all possess solutions illustrating waves and will be discussed in more detail in later sections.

Example 3.1. Suppose that pollutant is spilled into a fast moving stream. At a position x downstream from the spill, let $u(x,t)$ denote the concentration of pollutant in the water passing by at time t. Prior to the arrival of the polluted water, the value of $u(x,t)$ at position x is zero. As the pollutant carried by the stream passes by position x, the value of $u(x,t)$ increases and then decreases back to zero. The effect on the value of u due to the movement of the stream is called **advection** and is modeled by the **advection equation**

$$u_t + cu_x = 0.$$

The advection equation is also called the *transport* or *convection equation.*

Example 3.2. When pollutant is spilled into a still channel of water, the pollutant is no longer transported to other parts of the channel by the movement of water. In this case the main process for spreading the pollutant through the channel is diffusion. The **diffusion equation**

$$u_t = Du_{xx}$$

is derived as a basic model of this process. The spread of heat through a medium can also be a diffusive process and so the diffusion equation is also called the *heat equation.*

Example 3.3. The **linearized Burgers equation**

$$u_t + cu_x = Du_{xx}$$

is an equation which illustrates a combination of the transport and diffusion processes from the previous two examples. The **Burgers equation**

$$u_t + uu_x = Du_{xx}$$

is a fundamental equation from fluid mechanics that combines a different advection process with diffusion. When $D = 0$, the Burgers equation becomes the **inviscid Burgers equation**

$$u_t + uu_x = 0,$$

which provides a classic example of shock waves.

Example 3.4. The motion of a plucked guitar string is nearly perpendicular to the length of the string. The **wave equation**

$$u_{tt} = c^2 u_{xx}$$

is used to model the amount of displacement $u(x,t)$ at position x along the string at time t. Calling this equation *the* wave equation is not meant to suggest that this is the only equation which describes wave behavior.

Example 3.5. The **Korteweg-deVries equation**

$$u_t + uu_x + u_{xxx} = 0$$

was derived in 1895 by Korteweg and deVries to model waves on the surface of relatively shallow water. Of particular interest are solutions of this equation called *solitary waves* or *solitons*.

3.2. An intuitive view

The examples of partial differential equations given in the previous section are all derived from fundamental scientific principles. Many of these equations can be understood from an intuitive point of view by thinking of the physical and geometrical meanings of partial derivatives.

At a fixed position x in the medium, the value of $u_t(x,t)$ gives the rate at which u is changing with respect to time. A positive value of $u_t(x,t)$ indicates that at position x, u is increasing as time increases. A negative value of $u_t(x,t)$ indicates that u is decreasing at position x as time increases (see Figure 3.1). In applications where $u(x,t)$ denotes a displacement at position x, $u_t(x,t)$ and $u_{tt}(x,t)$ have the particular interpretation of velocity and acceleration.

At a fixed time t, the values of $u_x(x,t)$ and $u_{xx}(x,t)$ provide information about the slope and concavity of the graph of $u(x,t)$ as a function of x. In particular applications these represent quantities such as flux and stress.

Example 3.6. Suppose that $u(x,t)$ represents the temperature u at position x along a metal bar at time t. A snapshot of the temperature profile of the rod at time t is represented by the graph of $u(x,t)$ as a

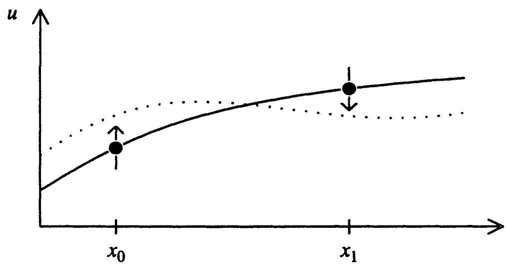


Figure 3.1. With respect to time, the value of u is increasing at position x_0 ($u_t(x_0, t) > 0$) and decreasing at position x_1 ($u_t(x_1, t) < 0$).

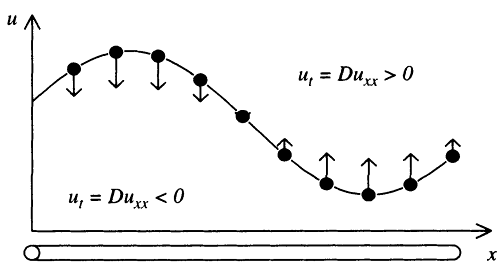


Figure 3.2. According to the heat equation, the change in temperature u at each point x depends on the concavity of the temperature profile at that instant.

function of x (see Figure 3.2). If the rod's temperature distribution is governed by the heat equation

$$u_t = Du_{xx} \quad (D > 0 \text{ constant}),$$

then the rate at which the temperature changes at position x is proportional to the concavity of the temperature profile at x. Since D is positive, the rate of change $u_t(x, t)$ at position x and the concavity $u_{xx}(x, t)$ will have the same sign. In particular, portions of the temperature profile $u(x, t)$ which are concave up ($u_{xx}(x, t) > 0$) correspond to points on the rod whose temperature will increase ($u_t(x, t) > 0$). Portions of the temperature profile which are concave down correspond to points on the rod whose temperature will

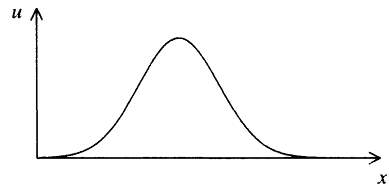


Figure 3.3. Initial profile $u(x, 0)$ for Exercise 3.7.

decrease. The larger (smaller) the magnitude of $u_{xx}(x, t)$, the greater the rate of increase (decrease) of u at position x (see Figure 3.2).

Exercise 3.7. Suppose the profile of $u(x, t)$ at time $t = 0$ is given by the graph shown in Figure 3.3. Assuming $u(x, t)$ satisfies the advection equation $u_t + u_x = 0$,

(a) Identify the points x in Figure 3.3 for which the value of u will decrease shortly after time $t = 0$.

(b) Identify the points x in Figure 3.3 for which the value of u will increase shortly after time $t = 0$.

(c) Based on (a) and (b), draw a rough sketch of how the profile of $u(x, t)$ might look at a time t shortly after $t = 0$.

Exercise 3.8. Suppose a string is stretched horizontally and then plucked. Let $u(x, t)$ represent the vertical displacement of the string at position x and time t.

(a) Give physical and graphical interpretations of the partial derivatives $u_t(x, t)$ and $u_{tt}(x, t)$. Give graphical interpretations of $u_x(x, t)$ and $u_{xx}(x, t)$.

(b) Suppose $u(x, t)$ satisfies the wave equation $u_{tt} = au_{xx}$ where a is a positive constant. What is an interpretation of the wave equation in terms of acceleration and concavity?

3.3. Terminology

Partial differential equations are described and classified using a number of different terms. While there is no classification scheme in which one type of equation is more likely to possess wave-like solutions than another, there is some general terminology which we will use.

The **order** of a partial differential equation is the order of the highest partial derivative appearing in the equation. First and second order equations are very common in applications since first and second derivatives have fundamental physical meanings.

Example 3.9. The Burgers equation $u_t + uu_x = Du_{xx}$ ($D > 0$ constant) is second order. The inviscid Burgers equation $u_t + uu_x = 0$ is first order.

A first order partial differential equation for $u(x, t)$ is called **first order linear** if it can be written in the form

$$a_1 u_t + a_2 u_x + bu = f \tag{3.1}$$

where a_1 and a_2 are not both zero, and a_1, a_2, b, and f are constants or functions of x and t (but do not depend on u). If a first order partial differential equation cannot be written in this form, then it is called **nonlinear**.

A second order partial differential equation for $u(x, t)$ is called **second order linear** if it can be written in the form

$$a_{11}u_{tt} + a_{12}u_{tx} + a_{22}u_{xx} + b_1 u_t + b_2 u_x + cu = f \tag{3.2}$$

where a_{11}, a_{12}, and a_{22} are not all zero, and a_{ij}, b_i, c, and f are constants or functions of x and t. If a second order equation cannot be put into this form, then it is called **nonlinear**.

The linear equations (3.1) and (3.2) are called **homogeneous** if the term $f = 0$; otherwise, they are said to be **nonhomogeneous**. Nonlinear equations are generally not classified as homogeneous or nonhomogeneous.

Example 3.10. The Burgers equation $u_t + uu_x = Du_{xx}$ ($D > 0$ constant) is nonlinear because the term uu_x is not in the form au_x where a is independent of u. The linearized Burgers equation $u_t + cu_x = Du_{xx}$ (c, D constants) is linear and homogeneous since it can be rewritten in the form (3.2) as

$$u_t + cu_x - Du_{xx} = 0.$$

Exercise 3.11. Give an example of a first order nonlinear partial differential equation for $u(x,t)$.

Exercise 3.12. For each of the following partial differential equations, (i) find its order and (ii) classify it as linear homogeneous, linear nonhomogeneous, or nonlinear. Assume c is a nonzero constant.

(a)	$u_t + cu_x = 0$	Advection equation
(b)	$u_t + cu_x = e^{-t}$	Advection with source term
(c)	$u_{tt} = c^2 u_{xx}$	Wave equation in one dimension
(d)	$u_{tt} - u_{xx} + u = 0$	Klein-Gordon equation
(e)	$u_{tt} - u_{xx} + \sin(u) = 0$	Sine-Gordon equation
(f)	$u_t + uu_x + u_{xxx} = 0$	Korteweg-deVries (KdV) equation

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Part 2

Traveling and Standing Waves

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Chapter 4

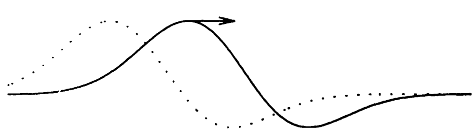
Traveling Waves

4.1. Traveling waves

One fundamental mathematical representation of a wave is

$$u(x,t) = f(x - ct)$$

where f is a function of one variable and c is a nonzero constant. The animation of such a function begins with the graph of the initial profile $u(x, 0) = f(x)$. If c is positive, then the profile of $u(x,t) = f(x - ct)$ at a later time t is a translation of the initial profile by an amount ct in the positive x direction. Such a function represents a disturbance moving with constant speed c:

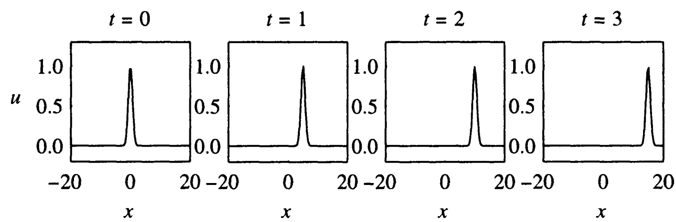


Similarly, $u(x,t) = f(x - ct)$ with $c < 0$ represents a disturbance moving in the negative x direction with speed $|c|$. In either case, the profile at each time t does not distort and remains a recognizable feature of a wave as it is translated along the x−axis.

Waves represented by functions of the form $u(x,t) = f(x - ct)$ are called **traveling waves**. The two basic features of any traveling wave are the underlying profile shape defined by f and the speed $|c|$

at which the profile is translated along the $x-$axis. It is assumed that the function f is not constant and c is not zero in order for $u(x,t)$ to represent the movement of a disturbance through a medium.

Example 4.1. The function $u(x,t) = e^{-(x-5t)^2}$ represents a traveling wave with initial profile $u(x,0) = e^{-x^2}$ moving in the positive x direction with speed 5. Four frames of the animation of this wave are



Example 4.2. The function $u(x,t) = \cos(2x + 6t)$ can be seen to represent a traveling wave by writing it as $u(x,t) = \cos[2(x+3t)]$. The initial profile $u(x,0) = \cos(2x)$ is being displaced in the *negative* x direction with a speed of 3.

A **traveling wave solution** of a partial differential equation is a solution of the differential equation which has the form of a traveling wave $u(x,t) = f(x - ct)$. Finding traveling wave solutions generally begins by assuming $u(x,t) = f(x-ct)$ and then determining which functions f and constants c yield a solution to the differential equation.

Example 4.3. Here we will find traveling wave solutions of the wave equation

$$u_{tt} = au_{xx}, \quad a > 0 \text{ constant.}$$

Assuming that $u(x,t) = f(x-ct)$, the chain rule gives

$$\begin{aligned} u_t(x,t) &= \left[f'(x-ct)\right](x-ct)_t = -cf'(x-ct), \\ u_x(x,t) &= \left[f'(x-ct)\right](x-ct)_x = f'(x-ct). \end{aligned}$$

Applying the chain rule a second time

$$\begin{aligned} u_{tt}(x,t) &= \left[-cf''(x-ct)\right](x-ct)_t = c^2 f''(x-ct), \\ u_{xx}(x,t) &= \left[f''(x-ct)\right](x-ct)_x = f''(x-ct) \end{aligned}$$

and substituting into the wave equation implies

$$c^2 f''(x - ct) = af''(x - ct).$$

Letting $z = x - ct$ and rearranging shows that we need to find c and $f(z)$ so that

$$(c^2 - a)f''(z) = 0$$

for all z.

One possibility is for $c^2 = a$. In this case f can be any twice differentiable function; taking any such nonconstant f and $c = \pm\sqrt{a}$, the two functions

$$u(x,t) = f(x - \sqrt{a}t), \quad u(x,t) = f(x + \sqrt{a}t)$$

are traveling wave solutions of the wave equation. Special examples include $u(x,t) = \sin(x - \sqrt{a}t)$, $u(x,t) = (x + \sqrt{a}t)^4$, and $u(x,t) = e^{-(x-\sqrt{a}t)^2}$. Another possibility is for $f'' = 0$, in which case f must be a linear function $f(z) = A + Bz$. The coefficient B should not be zero to ensure that the profile f is not constant. In this case

$$u(x,t) = A + B(x - ct)$$

is a traveling wave solution of the wave equation for any choice of A, B, c as long as $B \neq 0$ and $c \neq 0$.

Exercise 4.4. Find traveling wave solutions of the following equations.

(a) The advection equation $u_t + au_x = 0$ where a is a nonzero constant.

(b) The Klein-Gordon equation $u_{tt} = au_{xx} - bu$ where a and b are positive constants.

Exercise 4.5. Consider the Sine-Gordon equation $u_{tt} = u_{xx} - \sin u$.

(a) Show that the profile shape f of a traveling wave solution $u(x,t) = f(x - ct)$ of the Sine-Gordon equation must satisfy the differential equation

$$(1 - c^2)f''(z) = \sin(f(z))$$

where $z = x - ct$.

(b) The differential equation in part (a) is a second order nonlinear equation. Since this equation does not explicitly involve $f'(z)$, it can be reduced to a first order equation with the following technique. Multiply both sides of the differential equation in part (a) by $f'(z)$ and integrate both sides with respect to z to show that

$$(1-c^2)\left(f'(z)\right)^2 = A - 2\cos(f(z))$$

where A is an arbitrary constant of integration.

(c) In the special case $A = 2$ and $0 < c < 1$, show that the first order equation in part (b) can be rewritten in the form

$$\left(f'(z)\right)^2 = \frac{4}{1-c^2}\sin^2\left(f(z)/2\right).$$

Then verify that

$$f(z) = 4\arctan\left[\exp\left(\frac{z}{\sqrt{1-c^2}}\right)\right]$$

is a solution of this equation. Thus for any speed $0 < c < 1$,

$$u(x,t) = f(x-ct) = 4\arctan\left[\exp\left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right]$$

is a traveling wave solution of the Sine-Gordon equation.

Exercise 4.6. The previous exercise shows that

$$u(x,t) = 4\arctan\left[\exp\left(\frac{x-ct}{\sqrt{1-c^2}}\right)\right]$$

is a traveling wave solution of the Sine-Gordon equation for any speed $0 < c < 1$. Animate this traveling wave three times using three different choices of c. How does the profile of the traveling wave change with c?

4.2. Wave fronts and pulses

A sudden change in weather occurs when a cold front passes through a region. The temperature at points ahead of the front appear to be at a relatively constant k_1 degrees, while behind the disturbance the temperature has dropped (sometimes by more than 30°F) to a new temperature k_2. On a weather map, the sudden drop in temperature

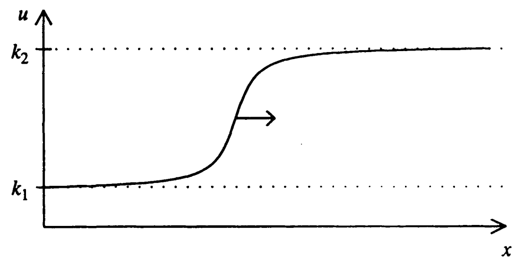


Figure 4.1. The profile of a wave front at time t.

is a recognizable feature which identifies the location and movement of this disturbance, so a cold front is an example of a wave.

A traveling wave such as the one profiled in Figure 4.1 is an example of a *wave front.* A traveling wave represented by $u(x,t)$ is said to be a **wave front** if for any fixed t,

$$u(x,t) \to k_1 \text{ as } x \to -\infty, \quad u(x,t) \to k_2 \text{ as } x \to \infty$$

for some constants k_1 and k_2. In general, the values k_1 and k_2 are not necessarily the same. In the particular case that the measure of u is approximately the same on both sides of the disturbance ($k_1 = k_2$), then the wave front is called a **pulse.** A pulse disturbance temporarily changes the value of u at position x before it settles back to its original value.

Example 4.7. The traveling wave $u(x,t) = e^{-(x-5t)^2}$ in Example 4.1 is a pulse since $\lim_{x\to\infty} e^{-(x-5t)^2} = 0$ and $\lim_{x\to-\infty} e^{-(x-5t)^2} = 0$. The traveling wave $u(x,t) = \cos(2x+6t)$ in Example 4.2 is not a wave front or a pulse since $\lim_{x\to\infty} u(x,t)$ does not exist.

Exercise 4.8. Is the traveling wave in Exercise 4.6 a wave front, pulse, or neither?

4.3. Wave trains and dispersion

The traveling wave $u(x,t) = \cos(2x+6t)$ from Example 4.2 is not a wave front or pulse, but rather an example of another type of wave.

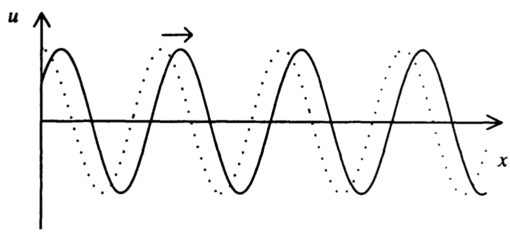


Figure 4.2. A wave train.

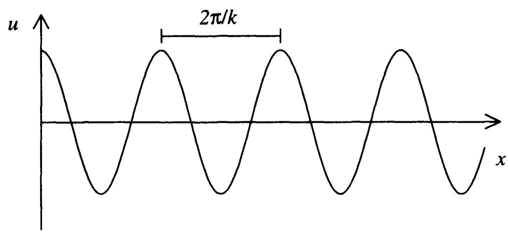


Figure 4.3. One cycle of a wave train.

A traveling wave which can be written in the form

$$u(x,t) = A\cos(kx - \omega t) \quad \text{or} \quad u(x,t) = A\cos(kx + \omega t)$$

where $A \neq 0$, $k > 0$ and $\omega > 0$ are constants is called a **wave train**. By rewriting $u(x,t)$ as

$$u(x,t) = A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]$$

one can see these are in fact traveling waves with profile shape $f(z) = A\cos(kz)$ moving with speed $c = \omega/k$ (see Figure 4.2). More generally, wave trains are represented as $u(x,t) = f(kx - \omega t)$ where $f(z)$ is a periodic function.

In a wave train $u(x,t) = A\cos(kx-\omega t)$, the number k is called the **wave number** and represents the number of cycles of this periodic wave that appear in a window of length 2π on the $x-$axis (Figure 4.3). The number ω is called the **circular frequency** and represents the number of cycles of the wave that pass by any fixed point x on the $x-$axis during a time interval of 2π.

A partial differential equation may have solutions which are wave trains, but not necessarily for every possible wave number k or frequency ω. To find which wave numbers and frequencies are permitted, one can substitute the form of a wave train such as $u(x,t) = A\cos(kx - \omega t)$ into the differential equation and reduce it to a relationship between k and ω. This relationship is called a **dispersion relation** and indicates which values of k and ω may be selected in order for $u(x,t)$ to be a wave train solution.

Example 4.9. Here we will look for wave train solutions of the form $u(x,t) = A\cos(kx - \omega t)$ for the advection equation

$$u_t + au_x = 0.$$

Computing the partial derivatives u_t and u_x of this wave train form shows $u(x,t)$ will be a solution of the advection equation if

$$\omega A\sin(kx - \omega t) + a\left[-kA\sin(kx - \omega t)\right] = 0,$$

or

$$A(\omega - ak)\sin(kx - \omega t) = 0.$$

The dispersion relation here is $\omega = ak$. Thus for any wave number k, $u(x,t) = \cos[k(x - at)]$ is a wave train solution traveling to the right with speed $c = a$.

Example 4.10. The Klein-Gordon Equation $u_{tt} = au_{xx} - bu$ (a, b positive constants) models the transverse vibration of a string with a linear restoring force. The wave train $u(x,t) = A\cos(kx - \omega t)$ is a solution of this equation if

$$-\omega^2 A\cos(kx - \omega t) = a\left[-k^2 A\cos(kx - \omega t)\right] - bA\cos(kx - \omega t)$$

or

$$A(\omega^2 - ak^2 - b)\cos(kx - \omega t) = 0.$$

Thus $u(x,t) = A\cos(kx - \omega t)$ is a solution of the Klein-Gordon equation if k and ω satisfy the dispersion relation $\omega^2 = ak^2 + b$. When $\omega = \sqrt{ak^2 + b}$, the wave train solution takes the traveling wave form

$$u(x,t) = A\cos\left(kx - \sqrt{ak^2 + b}\,t\right) = A\cos\left[k\left(x - \sqrt{\frac{ak^2 + b}{k^2}}\,t\right)\right]$$

with speed

$$c = \sqrt{\frac{ak^2 + b}{k^2}} = \sqrt{a + \frac{b}{k^2}} = \sqrt{a + \frac{ab}{\omega^2 - b}}. \tag{4.1}$$

There is a fundamental difference between the previous two examples. In the advection equation, all wave train solutions travel with the same speed $c = a$. In the Klein-Gordon example, equation (4.1) shows that wave trains with higher frequency ω travel with lower speed c. A partial differential equation which has wave train solutions is said to be **dispersive** if waves trains of different frequencies ω propagate through the medium with different speeds. The Klein-Gordon equation is dispersive while the advection equation is not.

Exercise 4.11. Suppose that waves in a medium are governed by the Klein-Gordon equation. Based on (4.1), what are the possible speeds that a wave train can move through the medium? In particular, how fast and how slow can a wave train move through the medium?

Exercise 4.12. In each of the following partial differential equations, find the dispersion relation for wave train solutions of the form $u(x,t) = A\cos(kx - \omega t)$, then determine if each equation is dispersive or not. Assume a is a positive constant.

(a)	$u_{tt} = au_{xx}$	The wave equation
(b)	$u_{tt} + au_{xxxx} = 0$	The beam equation
(c)	$u_t + u_x + u_{xxx} = 0$	The linearized KdV equation

Exercise 4.13. It is sometimes easier to find a dispersion relation using the complex wave train

$$u(x,t) = \cos(kx - \omega t) + i\sin(kx - \omega t) = e^{i(kx - \omega t)}$$

where i is the imaginary unit. In this case $u_x(x,t) = ike^{i(kx-\omega t)}$ and $u_t(x,t) = -i\omega e^{i(kx-\omega t)}$. Use this form of a wave train to find a dispersion relation for the following partial differential equations. Assume a and d are positive constants.

(a)	$u_t + au_x = du_{xx}$	The linearized Burgers equation
(b)	$iu_t + u_{xx} = 0$	The Schrödinger equation
(c)	$u_{tt} = au_{xx}$	The wave equation

Chapter 5

The Korteweg-deVries Equation

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded smooth and well-defined heap of water which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.[1]
>
> J.S. Russell, 1844.

[1] John Scott Russell, *Report on waves*, Report of the 14th Meeting of the British Association for the Advancement of Science, 1844, pp. 311–390.

5.1. The KdV equation

In 1834, J.S. Russell observed the phenomena of a large bulge of water slowly traveling along a channel of water. The ability of this water wave to retain its shape for such a long period of time was quite remarkable and led Russell to study this disturbance by conducting numerous detailed experiments. He later came to call this phenomena a Wave of Translation, highly suggestive of a traveling wave. Russell's work on the Wave of Translation is now considered the beginning study of what are now called *solitary waves* or *solitons.*

Russell's experiments and observations drew the attention of notable scientists such as Boussinesq, Rayleigh, and Stokes. In 1895, Korteweg and deVries derived a partial differential equation to model the height of the surface of shallow water in the presence of long gravity waves [**KdV**]. In these waves, the length of the wave is large compared to the depth of the water, as was the case in Russell's Wave of Translation. The differential equation of Korteweg and deVries,

$$U_t + (a_1 + a_2 U)U_x + a_3 U_{xxx} = 0, \quad a_2, a_3 \neq 0,$$

is a third order nonlinear equation now known as the Korteweg-deVries equation or **KdV equation**. A substitution of $u = a_1 + a_2 U$ and a scaling of the independent variables x and t results in the reduced form of the KdV equation,

$$\boxed{u_t + uu_x + u_{xxx} = 0.} \tag{5.1}$$

For a more complete reading on the history of solitons, the KdV equation, and other fundamental equations from which solitons arise, see the monograph *Solitons in Mathematics and Physics* by Alan C. Newell [**New**].

5.2. Solitary wave solutions

In this section we will look for traveling wave solutions of the reduced KdV equation (5.1). The solutions that will be found are called *solitons* and model the wave phenomena observed by Russell.

In the spirit of a "heap" of water forming a Wave of Translation, we will look for a traveling wave solution $u(x,t) = f(x - ct)$ in the form of a pulse, where $c > 0$, and $f(z)$, $f'(z)$, and $f''(z)$ tend to 0

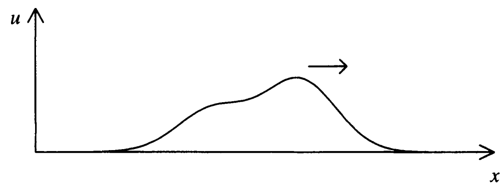


Figure 5.1. A pulse profile in which $u(x,t)$, $u_x(x,t)$, and $u_{xx}(x,t)$ approach 0 as $x \to \pm\infty$.

as $z \to \pm\infty$ (see Figure 5.1). Substituting $u(x,t) = f(x-ct)$ into the KdV equation $u_t + uu_x + u_{xxx} = 0$ forms a third order nonlinear ordinary differential equation for $f(z)$,

$$-cf' + ff' + f''' = 0.$$

This particular equation can be integrated once to get

$$-cf + \tfrac{1}{2}f^2 + f'' = a$$

where a is a constant of integration. From the assumptions that $f(z)$ and $f''(z) \to 0$ as $z \to \infty$, the value of a is zero. Multiplying by f'

$$-cff' + \tfrac{1}{2}f^2f' + f'f'' = 0$$

and integrating again results in the first order equation

$$-\tfrac{1}{2}cf^2 + \tfrac{1}{6}f^3 + \tfrac{1}{2}(f')^2 = b.$$

Since $f(z), f'(z) \to 0$ as $z \to \infty$ from the form of the pulse, the constant of integration b is zero. Solving for $(f')^2$ gives

$$3\left(f'\right)^2 = (3c - f)f^2.$$

From here we will require $0 < f(z) < 3c$ in order to have a positive right-hand side; taking the positive square root yields

$$\frac{\sqrt{3}}{\sqrt{3c-f}\,f}\,f' = 1.$$

To integrate the left hand side, make the rationalizing substitution $g^2 = 3c - f$; substituting $f = 3c - g^2$ and $f' = -2gg'$ results in

$$\frac{2\sqrt{3}}{3c - g^2}\,g' = -1.$$

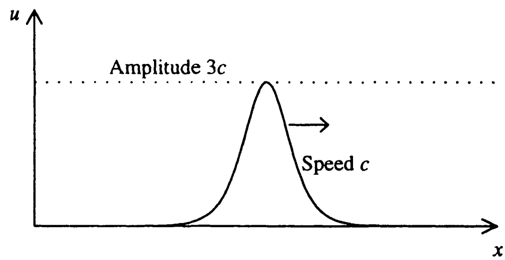


Figure 5.2. A profile of a solitary wave solution of the KdV equation.

By the method of partial fractions, integration of both sides with respect to z gives

$$\ln\left(\frac{\sqrt{3c}+g}{\sqrt{3c}-g}\right) = -\sqrt{c}z + d$$

for some constant of integration d. Solving for g yields

$$g(z) = \sqrt{3c}\,\frac{\exp(-\sqrt{c}z+d)-1}{\exp(-\sqrt{c}z+d)+1} = -\sqrt{3c}\tanh\left[\tfrac{1}{2}(\sqrt{c}z-d)\right],$$

and then computing $f = 3c - g^2$ results in

$$f(z) = 3c\,\mathrm{sech}^2\left[\tfrac{1}{2}(\sqrt{c}z-d)\right].$$

Recall that $\mathrm{sech}(z) = 1/\cosh(z)$, where $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$.

Since the arbitrary constant d is simply a shift of the shape

$$f(z) = 3c\,\mathrm{sech}^2\left[\tfrac{1}{2}\sqrt{c}z\right],$$

we can get a good idea of what this traveling wave looks like with $d = 0$. The resulting traveling wave solution to the KdV equation is

$$\boxed{u(x,t) = 3c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right].} \tag{5.2}$$

A profile of this wave is shown in Figure 5.2. This pulse is called a **solitary wave** or **soliton**.

Russell observed in his experiments that Waves of Translation with greater height moved with a greater velocity. This is borne out in the solution (5.2) of the KdV equation by observing that the amplitude of the wave is $3c$, three times the wave speed c.

Exercise 5.1. Find a pulse traveling wave solution of the *modified* KdV equation $u_t + u^2 u_x + u_{xxx} = 0$. This equation appears in electric circuit theory and multicomponent plasmas [**IR**].

Exercise 5.2. Since the KdV equation is nonlinear, the sum of two of its solutions is not necessarily another solution. To illustrate this, let v and w represent two solutions of $u_t + uu_x + u_{xxx} = 0$. Show that $u = v + w$ is a solution only when the product vw does not depend on x.

Exercise 5.3. (Interacting Solitary Waves) Suppose k_1 and k_2 are positive numbers and set

$$\begin{aligned} u_1(x,t) &= \exp(k_1^3 t - k_1 x), \\ u_2(x,t) &= \exp(k_2^3 t - k_2 x), \\ A &= (k_1 - k_2)^2/(k_1 + k_2)^2. \end{aligned}$$

Let

$$u = 12\,\frac{k_1^2 u_1 + k_2^2 u_2 + 2(k_1 - k_2)^2 u_1 u_2 + A u_1 u_2 (k_1^2 u_2 + k_2^2 u_1)}{(1 + u_1 + u_2 + A u_1 u_2)^2}.$$

This is a solution of $u_t + uu_x + u_{xxx} = 0$ derived using a method described in [**Whi**, pp. 580–583]. Taking $k_1 = 1$ and $k_2 = 2$, animate $u(x,t)$ for $-10 \le x \le 10$ and time $-10 \le t \le 10$ to observe the behavior of this double soliton solution. If animating using the script `wvmovie` provided with the companion MATLAB software (see page xiii), setting the $u(x,t)$ field to `kdv2(x,t)` will view this solution.

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Chapter 6

The Sine-Gordon Equation

In this chapter we will derive the Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

as a description of a *mechanical transmission line*, and look for traveling wave solutions of this equation.

6.1. A mechanical transmission line

In the late 1960's, A.C. Scott constructed a mechanical analogue of an electrical transmission line. This device consists of a series of pendula connected by a steel spring and supported horizontally by a thin wire (Figure 6.1). Each pendulum is free to swing in a plane perpendicular to the wire, however in doing so, the spring coils and provides a torque on the two neighboring pendula. This interaction between adjacent pendula permits a disturbance in one part of the device to propagate, mechanically transmitting a signal down the line of pendula. If a pendulum at one end of the device is disturbed slightly, then the transmitted disturbance results in a small "wavy" motion (Figure 6.2). A more dramatic effect occurs if a single pendulum at one end is quickly turned one full revolution around the wire (Figure 6.3).

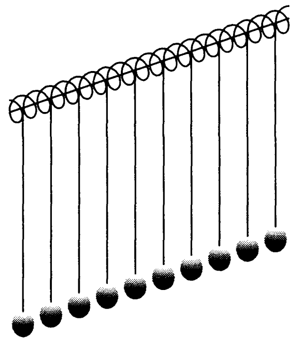


Figure 6.1. Pendula attached to a horizontal spring.

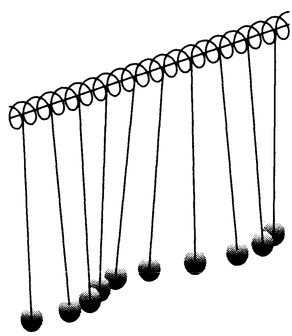


Figure 6.2. A small disturbance moving down the pendulum line.

6.2. The Sine-Gordon equation

In [**Sc**, pp. 48–49], the Sine-Gordon equation

$$\boxed{u_{tt} - u_{xx} + \sin u = 0} \tag{6.1}$$

is derived as a continuous model for describing motions of the pendula, where $u(x, t)$ represents the angle of rotation of the pendulum at position x and time t. The Sine-Gordon equation also arises in the study of superconductor transmission lines, crystals, laser pulses, and the geometry of surfaces. See [**Sc**, p. 250] for references.

We will now follow Scott's derivation of the Sine-Gordon equation (6.1) from a system of ordinary differential equations which models

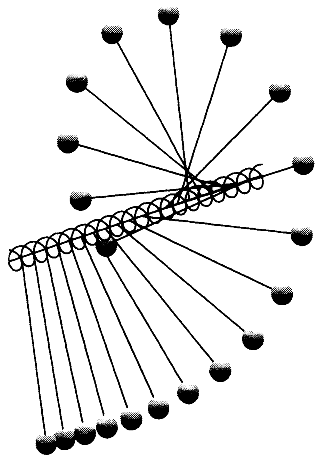


Figure 6.3. A large disturbance moving down the pendulum line.

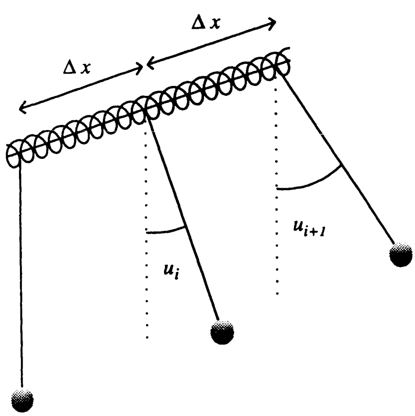


Figure 6.4

the angles of rotation of the pendula. Here we will assume that each pendulum has mass m and length l, and the pendula are equally spaced along the spring with a separation distance of Δx. Let $u_i(t)$ measure the angle of rotation of the i^{th} pendulum at time t, with $u_i = 0$ being the down position (Figure 6.4).

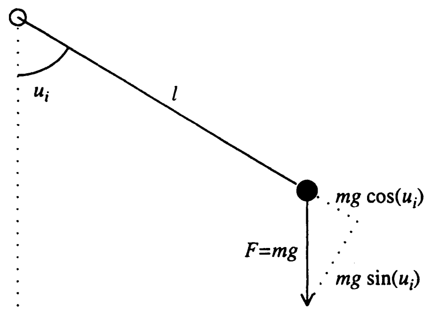


Figure 6.5

The mathematical model for the motion of the pendula is based on Newton's second law of motion in rotational form,

$$(6.2) \qquad I\frac{d^2u_i}{dt^2} = \text{net torque acting on the } i^{th} \text{ pendulum}$$

where I is the moment of inertia of the pendulum, $I = ml^2$, and torque is the measure of the turning effect of a force. In this case there are three torques which will be taken into account—the torque due to gravity, the torque due to the twisting of the spring coiled between pendula i and $(i-1)$, and the torque due to the spring between pendula i and $(i+1)$.

Looking first at torque due to gravity, the gravitational force acting on the i^{th} pendulum tries to rotate the pendulum downward. As shown in Figure 6.5, the resulting torque is $\pm(mg\sin u_i)(l)$, where $mg\sin u_i$ is the amount of gravitational force perpendicular to the pendulum, l is the distance from the pivot point to the mass, and g is the acceleration due to gravity. Figure 6.5 also indicates the sign (direction) of the torque. If the pendulum has swung to the right $(0 < u_i < \pi/2)$, then $\sin u_i > 0$. The gravitational torque, however, will try to rotate the pendulum back to the left in the negative u_i direction. If the pendulum has swung to the left $(-\pi/2 < u_i < 0)$, then $\sin u_i < 0$. The gravitational torque, however, will try to rotate the pendulum back to the right in the direction of positive u_i. The portion of the net torque due to gravity is then $-mgl\sin u_i$ to account for the correct sign.

The next torque to be accounted for is the turning effect due to the portion of the spring between the i and $(i+1)$ pendula. Intuition suggests the strength of this turning effect depends on three major factors—the amount of twisting of the spring, the length of that part of the spring, and the stiffness of the spring's material. One model for this torque is

$$\text{Spring torque} = K\frac{u_{i+1} - u_i}{\Delta x}$$

where $u_{i+1} - u_i$ is the amount of twist in the part of the spring between the i and $(i+1)$ pendula, Δx is the length of that part of the spring, and $K > 0$ is a spring constant depending upon the spring's material. If $u_{i+1} - u_i = 0$, then both ends of that segment of spring have been rotated the same amount and so no twisting between pendula i and $(i+1)$ has taken place. Large values of $u_{i+1} - u_i$ correspond to one end of this segment of spring being rotated much more than the other, coiling the spring and resulting in a large torque on the i^{th} pendulum. Long sections of spring (large Δx) result in smaller torques since there are more coils of the spring to absorb twisting of the spring.

Similarly, the torque applied to the i^{th} pendulum due to the twisting of the spring between the i and $(i-1)$ pendula will be assumed to be

$$K\frac{(u_{i-1} - u_i)}{\Delta x}.$$

Putting the gravitation and spring torques in Newton's second law (6.2) results in

$$ml^2\frac{d^2u_i}{dt^2} = K\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x} - mgl\sin u_i. \tag{6.3}$$

Now suppose the number of pendula is increased while decreasing their mass in such a way that $m/\Delta x \to M$ as $\Delta x \to 0$. This forms a continuous "sheet" of material with mass density M. Let $u(x,t)$ denote the angle of rotation of this continuous sheet at position x and time t. Dividing (6.3) by another factor of Δx gives

$$\frac{ml^2}{\Delta x}\frac{d^2u_i}{dt^2} = K\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \frac{mgl}{\Delta x}\sin u_i,$$

and so taking the limit $\Delta x \to 0$ results in

$$Ml^2 u_{tt} = K u_{xx} - Mgl \sin u.$$

Setting $A = Ml^2$ and $T = Mgl$ puts this in the form of the Sine-Gordon equation

$$\boxed{A u_{tt} - K u_{xx} + T \sin u = 0.} \tag{6.4}$$

Exercise 6.1. The more general Sine-Gordon equation (6.4) can be reduced to the form (6.1) through a change of independent variables. Suppose $u(x,t)$ is a solution of

$$A u_{tt} - K u_{xx} + T \sin u = 0.$$

Let ξ and τ be a new set of independent variables formed by the scaling $\xi = ax$ and $\tau = bt$. Letting $U(\xi,\tau)$ be defined by $U(\xi,\tau) = u(x,t)$, find scaling constants a and b so that $U(\xi,\tau)$ is a solution of

$$U_{\tau\tau} - U_{\xi\xi} + \sin U = 0.$$

Exercise 6.2. The Sine-Gordon equation $u_{tt} - u_{xx} + \sin u = 0$ is a special case of the more general form $u_{tt} - u_{xx} + V'(u) = 0$ where $V(u)$ represents potential energy. What is the potential energy function $V(u)$ for the Sine-Gordon equation?

6.3. Traveling wave solutions

In this section we will look for traveling wave solutions of the Sine-Gordon equation (6.1),

$$u_{tt} - u_{xx} + \sin u = 0.$$

Letting $u(x,t) = f(x - ct)$ and substituting into the Sine-Gordon equation gives

$$c^2 f'' - f'' + \sin f = 0.$$

The equation formed after multiplying by f',

$$(c^2 - 1) f'' f' + (\sin f) f' = 0,$$

can be integrated to produce the first order equation

$$\tfrac{1}{2}(c^2 - 1)(f')^2 - \cos f = a.$$

Additional conditions are needed to find the constant of integration a. With an eye towards the pendulum problem, we will look for a solution f which satisfies $f(z) \to 0$ and $f'(z) \to 0$ as $z \to \infty$ to approximate the notion of undisturbed pendula ahead of a moving disturbance. In this case $a = -1$, so

$$(f')^2 = \frac{2}{1-c^2}(1-\cos f) = \frac{4}{1-c^2}\sin^2(f/2).$$

Here the speed c of the traveling wave will need to satisfy $c^2 < 1$ to ensure that the right hand side is positive. One solution of this equation is (see Exercise 4.5)

$$f(z) = 4\arctan\left[\exp\left(-\frac{z}{\sqrt{1-c^2}}\right)\right],$$

resulting in the traveling wave solution

$$\boxed{u(x,t) = 4\arctan\left[\exp\left(-\frac{x-ct}{\sqrt{1-c^2}}\right)\right].}$$

Four frames of animation of this traveling wave are shown in Figure 6.6. Since $u(x,t) \to 0$ as $x \to \infty$ and $u(x,t) \to 2\pi$ as $x \to -\infty$, this traveling wave is a wave front. Ahead of the wave front the pendula are in their undisturbed state (angle u near 0) while behind the wave front the pendula are near an angle of 2π, indicating that these pendula have rotated completely around the horizontal spring exactly once.

Exercise 6.3. Locate a traveling wave solution $u(x,t) = f(x-ct)$ of $u_{tt} - u_{xx} + \sin u = 0$ where $f(z) \to \pi$ and $f'(z) \to 0$ as $z \to \infty$. In terms of the pendula problem, what is a physical interpretation of this solution?

Exercise 6.4. Verify by direct substitution that the following is a solution of $u_{tt} - u_{xx} + \sin u = 0$:

$$u(x,t) = 4\arctan\left[\frac{\sinh(ct/\sqrt{1-c^2})}{c\cosh(x/\sqrt{1-c^2})}\right].$$

Animate this solution with $c = 1/2$, $-20 \le x \le 20$, and $-50 \le t \le 50$. This solution is called a *particle-antiparticle* collision [**PS**].

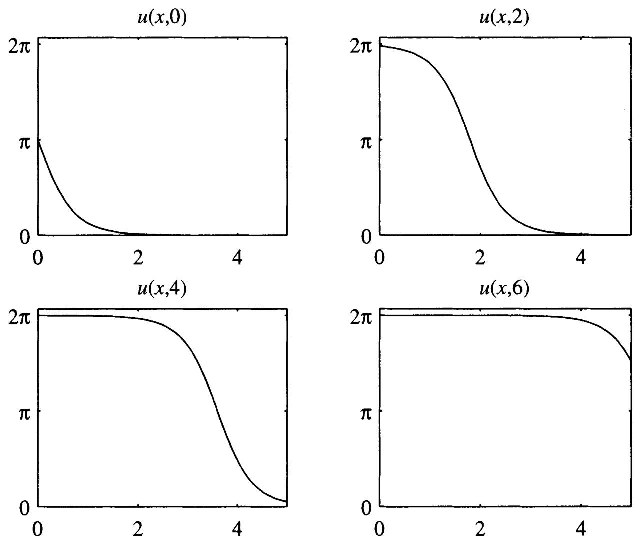


Figure 6.6. A Sine-Gordon traveling wave.

Exercise 6.5. If the motion of the pendula sheet is small (angle u remains close to zero), then one may make the approximation $\sin u \approx u$ in the Sine-Gordon equation (6.4). This results in a linear equation,

$$Au_{tt} - Ku_{xx} + Tu = 0.$$

This equation is called the Klein-Gordon equation.

(a) Find all traveling wave solutions for this linear equation.

(b) If the motion of the pendula is indeed small, then u must remain bounded. Which speeds c admit a traveling wave solution which is bounded?

(c) The bounded traveling wave solutions from part (b) are wave trains. Is the Klein-Gordon equation dispersive? In particular, do wave train solutions with high frequency travel with faster, slower, or same speed as solutions with low frequency?

(d) Show that there is a cutoff frequency ω_0 such that solutions with frequency $\omega \leq \omega_0$ are not permitted.

Chapter 7

The Wave Equation

In this chapter the wave equation $u_{tt} = c^2 u_{xx}$ is introduced as a model for the vibration of a stretched string.

7.1. Vibrating strings

The wave equation $u_{tt} = c^2 u_{xx}$ is a fundamental equation which describes wave phenomena in a number of different settings. One basic use of the wave equation is to model small vibrations, such as those of a plucked guitar string. In this section, assumptions are made about the way in which such a string vibrates. In the following section, these assumptions are used to derive the wave equation.

Suppose a string is initially stretched between two posts with the equilibrium position of the string lying along the x axis. After the string is plucked, let $u(x, t)$ measure the displacement of the string at position x and time t (Figure 7.1). The values of $u_t(x, t)$ and $u_{tt}(x, t)$ represent the vertical velocity and acceleration of the point on the string at position x. The derivative $u_x(x, t)$ measures the slope of the string at position x.

The way in which a string vibrates depends on properties of the string as well as any forces that are present. The following assumptions about a string will be used in the derivation of the wave equation.

- *Uniform string.* The string has a constant density ρ (mass per unit length).

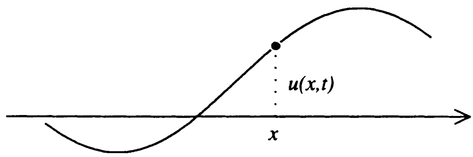
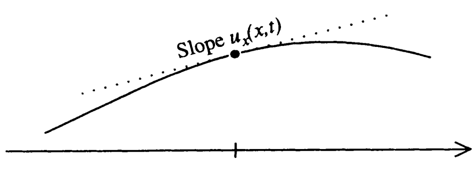


Figure 7.1. At time t, $u(x,t)$ is the displacement of the string at position x.

- *Planar vibrations.* The string remains in a plane as it vibrates.
- *Uniform tension.* A flexible connector such as a piece of rope or string exerts a force only in a direction parallel to itself. Such a force, called *tension*, acts as a pull on whatever is attached to the end of the string. We will assume that our vibrating string has constant tension—each piece of the string pulls on its neighboring segments of string with the same magnitude of force T. The direction of this force, however, varies and is tangent to the string at each point.
- *No other forces.* The only force present which affects the motion of the string is tension. Gravitational, frictional, magnetic, and other external forces will be omitted for now.
- *Small vibrations.* As the string vibrates, the slope $u_x(x,t)$ at each point of the string remains small:



7.2. A derivation of the wave equation

The wave equation will now be derived by applying Newton's Second Law of Motion to a piece of the string. Let S represent the segment of the string between x and $x + \Delta x$, where $\Delta x > 0$ is small (Figure 7.2). Thinking of this small piece of string as a body which moves only in

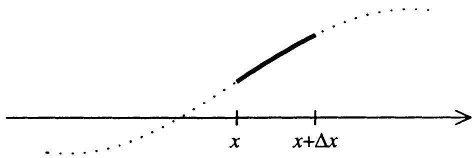


Figure 7.2. A piece S of the string.

the vertical direction, Newton's Second Law gives

$$\text{(Mass of } S)(\text{Acceleration of } S) = \text{Net force acting on } S \tag{7.1}$$

where acceleration and force are in a direction perpendicular to the $x-$axis. The next step is to calculate the mass, acceleration, and net force acting on S.

The mass of S is the string density ρ times the length of S, so

$$\text{Mass of } S = \rho \cdot \int_x^{x+\Delta x} \sqrt{1 + (u_x(s,t))^2}ds.$$

The value of $|u_x|$ is close to 0 under the assumption of small vibrations, so $\sqrt{1 + (u_x)^2}$ is approximately 1 if $(u_x)^2$ is assumed to be significantly smaller than one. In this case the mass of S will be approximated by

$$\text{Mass of } S \approx \rho \int_x^{x+\Delta x} 1 \; ds = \rho\Delta x. \tag{7.2}$$

Next, the vertical acceleration of S is $u_{tt}(x,t)$ if one chooses the left end of x to identify the position of the segment S.

Finally, it was assumed that tension is the only force present which acts on the string, so the net force applied to S is the net tension acting on S. The net tension here is the result of the pulling on the ends of S by the portions of the string to its right and left (Figure 7.3). On the left end of S, the tension force pulls left with magnitude T in a direction parallel to the string. The vector $-(1, u_x(x,t))$ is a left-pointing vector tangent to the string at position x (see Figure 7.4); dividing by its length constructs a unit tangent vector. Multiplying the unit vector by T then forms the left-pointing tension force

$$-T\frac{(1, u_x(x,t))}{\sqrt{1 + (u_x(x,t))^2}}$$

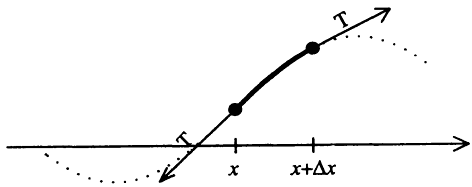


Figure 7.3. Tension forces pulling on string segment S.

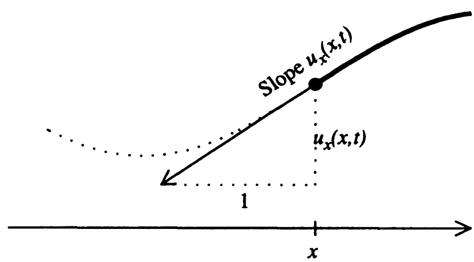


Figure 7.4. The left-pointing tangent vector $-(1, u_x)$.

acting on the left end of S. The vertical component of this force is then

$$-T\frac{u_x(x,t)}{\sqrt{1+(u_x(x,t))^2}}.$$

Under the assumption of small vibrations, we again make the approximation $\sqrt{1+(u_x)^2} \approx 1$, and so the vertical component of the force due to tension on the left side of S is approximately

$$-Tu_x(x,t).$$

Repeating this construction at the right end $x + \Delta x$ of S, the vertical component of the force due to tension on the right side of S is approximately

$$Tu_x(x+\Delta x,t).$$

The net vertical force acting on S is then given by

(7.3) $$\text{Net force on } S = Tu_x(x+\Delta x,t) - Tu_x(x,t).$$

Substituting the approximations for the mass (7.2), acceleration $u_{tt}(x,t)$, and force (7.3) into Newton's Second Law (7.1) gives

$$(\rho\Delta x)u_{tt}(x,t) = Tu_x(x+\Delta x,t) - Tu_x(x,t). \tag{7.4}$$

Dividing by Δx produces the form

$$\rho u_{tt}(x,t) = T\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x},$$

which by letting $\Delta x \to 0$ results in

$$\rho u_{tt}(x,t) = Tu_{xx}(x,t).$$

Setting $c = \sqrt{T/\rho}$ gives the common form of the wave equation,

$$\boxed{u_{tt} = c^2 u_{xx}.} \tag{7.5}$$

Exercise 7.1. A more general equation for the string is

$$\rho u_{tt} = Tu_{xx} - Fu_t - Ru + f(x,t)$$

where F and R are nonnegative constants. The additional terms on the right side represent additional forces acting on the string:

$$\begin{aligned} -Fu_t &= \text{force due to friction},\\ -Ru &= \text{linear restoring force},\\ f(x,t) &= \text{external force such as gravity}. \end{aligned}$$

Explain how the force $-Ru$ affects the string when $u(x,t) > 0$ and $u(x,t) < 0$. Explain why the force $-Fu_t$ is opposite in sign from the vertical velocity u_t.

Exercise 7.2. Suppose that the string is not homogeneous, that is, the tension and density of the string vary along the string as functions $T = T(x)$ and $\rho = \rho(x)$. Ignoring all other forces, follow the derivation for the homogeneous string to show that

$$\rho(x)u_{tt} = (T(x)u_x)_x.$$

(The mass of segment S is now $\int_x^{x+\Delta x} \rho(s)\sqrt{1+(u_x(s,t))^2}ds$. Approximate the density of segment S with $\rho(s) \approx \rho(x)$ for $x \le s \le x + \Delta x$.)

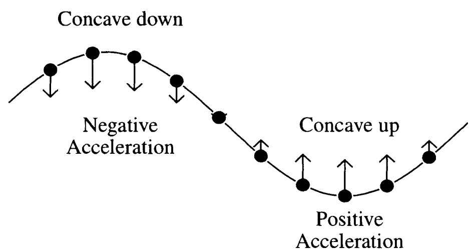


Figure 7.5. Graphical interpretation of solutions of $u_{tt} = c^2 u_{xx}$.

7.3. Solutions of the wave equation

Some general remarks about solutions of the wave equation are given in this section. The following chapters will look more closely at constructing particular solutions.

It follows from Example 4.3 of Section 4.1 that if $f(z)$ is any nonconstant twice differentiable function, then $u(x,t) = f(x - ct)$ and $u(x,t) = f(x + ct)$ are traveling wave solutions of the wave equation (7.5). This shows that the value of c in $u_{tt} = c^2 u_{xx}$ is the speed at which any traveling wave will propagate along the string. Since $c = \sqrt{T/\rho}$, increasing the tension T in a string will increase the speed at which traveling waves move along the string. Strings made of dense materials (larger values of ρ) have slow moving traveling waves.

A graphical interpretation of $u_{tt} = c^2 u_{xx}$ is that the vertical acceleration u_{tt} at a point on the string is proportional to the concavity u_{xx} of the string at that point. Portions of the string which are concave up are accelerating upward, while portions of the string which are concave down are accelerating downward (Figure 7.5).

While the wave equation $u_{tt} = c^2 u_{xx}$ is only one of many equations which admit wave-like solutions, it does describe a number of physical phenomena. Some other examples besides a vibrating string include longitudinal vibrations of a long slender bar, and current and voltage in electrical transmission lines.

Exercise 7.3. The following problem is an example of an *Initial Boundary Value Problem* for the wave equation:

$$\begin{aligned} \text{PDE:} \quad & u_{tt} = u_{xx}, \quad 0 < x < 1,\ 0 < t < \infty, \\ \text{BC:} \quad & u(0,t) = \sin(t), \quad u(1,t) = 0, \\ \text{IC:} \quad & u(x,0) = 0, \quad u_t(x,0) = 0. \end{aligned}$$

The conditions BC are called *boundary conditions* and the conditions IC are called *initial conditions.* In terms of a vibrating string, give a physical interpretation for each part of this initial boundary value problem. Based on this interpretation, sketch several frames of animation which might represent the solution $u(x,t)$ for a short period of time starting at $t = 0$.

Exercise 7.4. (Transmission Line Equations) Co-axial cables are often used in audio and video applications. At time t, let $i(x,t)$ and $v(x,t)$ denote the current and voltage (respectively) at position x along a co-axial cable.

The *transmission line equations* approximating the current and voltage in the cable are

$$\begin{aligned} i_x + Cv_t + Gv &= 0, \\ v_x + Li_t + Ri &= 0 \end{aligned}$$

where C is the capacitance per unit length of the cable, G is the leakage per unit length, R is the resistance per unit length, and L is the inductance per unit length.

(a) Show that if C, G, R, and L are constants, then eliminating v in the transmission line equations results in

$$i_{xx} = (CL)i_{tt} + (CR + GL)i_t + (GR)i.$$

(Differentiate the first equation with respect to x and the second equation with respect to t, combine the results to eliminate v_{tx} and v_{xt}, and then use the second equation $v_x + Li_t + Ri = 0$ again to eliminate the remaining v_x term.)

(b) Show that in a similar manner, the transmission line equations can be combined to eliminate i.

If $R = 0$ and $G = 0$, the two second order equations derived above give wave equations for the current and voltage

$$i_{tt} = c^2 i_{xx}, \quad v_{tt} = c^2 v_{xx}$$

with $c = \sqrt{1/(CL)}$.

Chapter 8

D'Alembert's Solution of the Wave Equation

In this chapter it will be shown that the general solution of the wave equation $u_{tt} = c^2 u_{xx}$ is the sum of two traveling waves, one moving right and the other moving left:

$$u(x,t) = F(x - ct) + G(x + ct).$$

This form will then be used to show that the solution of the initial value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x) \end{aligned}$$

can be written as

$$u(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

8.1. General solution of the wave equation

Recall from Example 4.3 that the wave equation $u_{tt} = c^2 u_{xx}$ admits traveling wave solutions of the form $h(x-ct)$ and $h(x+ct)$. There may be, however, other solutions which are not in the form of a traveling wave. In this section we will derive the general solution of the wave equation and show that it is the sum of *two* traveling waves.

Since traveling waves $h(x - ct)$ and $h(x + ct)$ are already known to be solutions of the wave equation, the change of variables

$$\xi = x - ct, \quad \eta = x + ct$$

will be made to establish a coordinate system which "follows" traveling waves to the left and right. In this new coordinate system the solution of the wave equation is more easily constructed.

Let $\xi(x,t) = x - ct$, $\eta(x,t) = x + ct$ and define $U(\xi, \eta)$ by

$$u(x,t) = U(\xi(x,t), \eta(x,t)).$$

Then by the multivariate chain rule,

$$\begin{aligned}
u_t &= U_\xi \xi_t + U_\eta \eta_t = -cU_\xi + cU_\eta, \\
u_{tt} &= -c(U_{\xi\xi}\xi_t + U_{\xi\eta}\eta_t) + c(U_{\eta\xi}\xi_t + U_{\eta\eta}\eta_t) \\
&= -c(-cU_{\xi\xi} + cU_{\xi\eta}) + c(-cU_{\eta\xi} + cU_{\eta\eta}) \\
&= c^2 U_{\xi\xi} - 2c^2 U_{\xi\eta} + c^2 U_{\eta\eta}, \\
u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta, \\
u_{xx} &= (U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x) + (U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x) \\
&= (U_{\xi\xi} + U_{\xi\eta}) + (U_{\eta\xi} + U_{\eta\eta}) \\
&= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}.
\end{aligned}$$

Substituting these expressions for u_{tt} and u_{xx} into $u_{tt} = c^2 u_{xx}$ and simplifying results in a relatively simple partial differential equation for $U(\xi, \eta)$,

$$U_{\xi\eta} = 0.$$

Writing this as $(U_\xi)_\eta = 0$ implies U_ξ is constant with respect to η, and so U_ξ is a function of ξ,

$$U_\xi = \phi(\xi).$$

Integrating with respect to ξ gives

$$U(\xi, \eta) = \int \phi(\xi) d\xi + G(\eta) = F(\xi) + G(\eta)$$

where $G(\eta)$ is a constant of integration with respect to ξ and $F(\xi)$ is an anti-derivative of $\phi(\xi)$. Converting ξ and η back to x and t shows that the general solution of $u_{tt} = c^2 u_{xx}$ is

$$\boxed{u(x,t) = F(x - ct) + G(x + ct)} \tag{8.1}$$

where F and G can be *any* two twice-differentiable functions. The general solution $u(x,t)$ of the wave equation is the sum of two traveling waves with speed c, one moving right and the other moving left.

Example 8.1. The following are all solutions of $u_{tt} = c^2 u_{xx}$:

$$\begin{aligned} u(x,t) &= e^{x-ct}, \\ u(x,t) &= \sin(x + ct), \\ u(x,t) &= (x - ct)^2 + e^{-(x+ct)^2}. \end{aligned}$$

The first two are traveling waves; however, the third is not. The first solution is purely a right moving wave $F(x - ct) = e^{x-ct}$ with no left moving part. The second solution consists only of a left moving wave $G(x + ct) = \sin(x + ct)$, while the third solution is a combination of left and right moving waves.

Exercise 8.2. Use direct substitution to verify that the function $u(x,t) = \cos(t)\sin(x)$ is a solution of $u_{tt} = u_{xx}$. According to the general form (8.1) for solutions of the wave equation, it must then be possible to write $u(x,t)$ as $u(x,t) = F(x - t) + G(x + t)$. What are the left and right moving waves in this case?

8.2. The d'Alembert form of a solution

When given the acceleration of an object moving along a line, one can recover the position of the object by integrating twice. This results in two constants of integration which are determined by the initial position and initial velocity of the object. Similarly with the vibrating string, we will now assume that the initial position $u(x,0)$ and initial velocity $u_t(x,0)$ are given for all values of x along the string. For example, a string which is initially at rest has initial position $u(x,0) = 0$ and velocity $u_t(x,0) = 0$. A string which is given an initial pluck by pulling on the string and simply letting go has an initial profile shape $u(x,0) = f(x)$ and velocity $u_t(x,0) = 0$.

In this section we will construct the solution of the following initial value problem for the displacement $u(x,t)$ of an infinitely long vibrating string:

$$\begin{aligned} \text{PDE:}\quad & u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty,\ t > 0, \\ \text{IC:}\quad & u(x,0) = f(x), \\ & u_t(x,0) = g(x). \end{aligned}$$

The wave equation (PDE) describes the acceleration of the string, while the initial conditions (IC) give the initial position and velocity of the string in terms of given functions $f(x)$ and $g(x)$.

To solve this problem, one can start with the general form for solutions of the wave equation

$$u(x,t) = F(x-ct) + G(x+ct).$$

Substituting into the initial position condition $u(x,0) = f(x)$ gives

$$F(x) + G(x) = f(x), \tag{8.2}$$

while substituting into the initial velocity condition $u_t(x,0) = g(x)$ results in

$$-cF'(x) + cG'(x) = g(x).$$

Dividing this last equation by c and integrating from 0 to x provides us with a second equation for $F(x)$ and $G(x)$,

$$-F(x) + G(x) = -F(0) + G(0) + \frac{1}{c}\int_0^x g(s)ds. \tag{8.3}$$

The conditions (8.2) and (8.3) form a system of two linear equations for $F(x)$ and $G(x)$; solving this system for $F(x)$ and $G(x)$ gives

$$\begin{aligned} F(x) &= \frac{1}{2}f(x) - \frac{1}{2}(-F(0) + G(0)) - \frac{1}{2c}\int_0^x g(s)ds, \\ G(x) &= \frac{1}{2}f(x) + \frac{1}{2}(-F(0) + G(0)) + \frac{1}{2c}\int_0^x g(s)ds. \end{aligned}$$

Once $F(x)$ and $G(x)$ have been found, the solution $u(x,t)$ of the initial value problem is then constructed by computing $F(x-ct)+G(x+ct)$,

$$\begin{aligned}
u(x,t) &= F(x-ct)+G(x+ct)\\
&= \frac{1}{2}f(x-ct)-\frac{1}{2}(-F(0)+G(0))-\frac{1}{2c}\int_0^{x-ct} g(s)ds\\
&+ \frac{1}{2}f(x+ct)+\frac{1}{2}(-F(0)+G(0))+\frac{1}{2c}\int_0^{x+ct} g(s)ds\\
&= \frac{1}{2}\left(f(x-ct)+f(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.
\end{aligned}$$

The resulting form of the wave equation solution given by

$$\boxed{u(x,t)=\frac{1}{2}\left(f(x-ct)+f(x+ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds} \tag{8.4}$$

is called the **d'Alembert solution** to the wave equation and explicitly gives u in terms of the initial information $f(x)$ and $g(x)$. The wave equation is unique: rarely can one find such an explicit form for solutions of a partial differential equation.

Example 8.3. The solution of the initial value problem

$$\begin{aligned}
u_{tt} &= c^2u_{xx}, \quad -\infty < x < \infty,\ t>0,\\
u(x,0) &= e^{-x^2},\\
u_t(x,0) &= 0
\end{aligned}$$

can be found using (8.4) with $f(x)=e^{-x^2}$ and $g(x)=0$,

$$u(x,t)=\tfrac{1}{2}\left(e^{-(x-ct)^2}+e^{-(x+ct)^2}\right).$$

The initial shape $u(x,0)=e^{-x^2}$ splits into two traveling waves, each half as high as the original shape and traveling in opposite directions. Four frames of animation with $c=1$ are shown in Figure 8.1.

Exercise 8.4. Consider the initial value problem

$$\begin{aligned}
u_{tt} &= c^2u_{xx}, \quad -\infty < x < \infty,\ t>0,\\
u(x,0) &= \sin(x),\\
u_t(x,0) &= 0.
\end{aligned}$$

(a) Find the d'Alembert solution of this initial value problem.

(b) Animate the solution from part (a) with $c=1$.

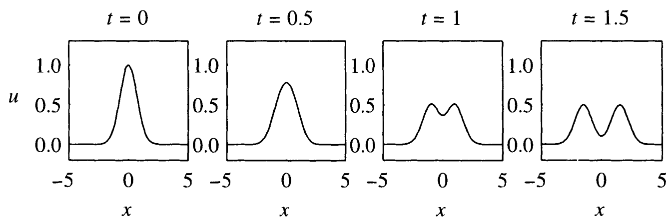


Figure 8.1. Profiles of the solution of the wave equation with initial profile $u(x,0) = e^{-x^2}$.

(c) Recall that a solution of the wave equation is really the sum of two traveling waves. To study the solution from part (a) in more detail, split your solution from part (a) into a right-traveling wave $F(x - ct)$ and a left-traveling wave $G(x + ct)$. Then animate $F(x-ct)$ and $G(x+ct)$ separately. The solution $u(x,t)$ is the sum of these two traveling waves.

(d) For another view of the solution, use trigonometric identities to write the solution in (a) as $u(x,t) = \cos(ct)\sin(x)$. This shows that the animation of $u(x,t)$ looks like the profile shape $\sin(x)$ with amplitude $\cos(ct)$ oscillating between -1 and 1.

Exercise 8.5. Find the d'Alembert solution of the following initial value problem and animate the result:

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= xe^{-x^2}. \end{aligned}$$

Exercise 8.6. Find the solution of the following initial value problem:

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \sin(x), \\ u_t(x,0) &= xe^{-x^2}. \end{aligned}$$

How does this problem and its solution relate to the previous two exercises?

Chapter 9

Vibrations of a Semi–infinite String

In the previous chapter, solutions of the wave equation were constructed to describe the motion of an infinite string extending over the entire x axis. A string that has one end and extends indefinitely in only one direction is called **semi-infinite**. In this chapter we will construct solutions of the wave equation to describe the motion of semi-infinite strings in two settings—when the end of the string is fastened down, and when the end of the string is free to move. Of particular interest will be the behavior of a disturbance, propagating along a semi-infinite string, when it encounters the end of the string.

9.1. A semi-infinite string with fixed end

Suppose that a string is semi-infinite, extending over the region $0 \leq x < \infty$, with the end of the string at $x = 0$ pinned down (Figure 9.1). In this case the displacement $u(x,t)$ at $x = 0$ is 0 for all t, forming

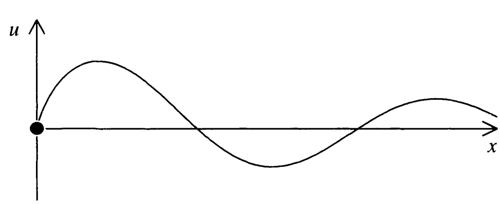


Figure 9.1. Semi-infinite string with left end fixed.

an equation $u(0,t) = 0$ called a **boundary condition**. If the initial position and velocity of the string are given, then the wave equation together with the boundary and initial conditions form an initial boundary value problem for the motion of this semi-infinite string:

$$\text{(9.1)}\qquad \begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0, && \text{(PDE)}\\ u(x,0) &= f(x), && \text{(IC)}\\ u_t(x,0) &= g(x), &&\\ u(0,t) &= 0. && \text{(BC)} \end{aligned}$$

The method used in Section 8.2 to derive the d'Alembert solution of the wave equation will now be used to construct a solution of this problem.

Recall from Section 8.2 that the general solution of the wave equation $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = F(x-ct) + G(x+ct),$$

where the functions F and G are constructed from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ in (9.1) by

$$\begin{aligned} F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds,\\ G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds. \end{aligned}$$

Note that here $f(x)$ and $g(x)$ are defined only for $x \geq 0$ since the string is semi-infinite. The resulting form of the wave equation solution

$$\text{(9.2)}\qquad u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

is valid as long as $x - ct$ and $x + ct$ are nonnegative. Assuming that c is positive, the value of $x + ct$ is nonnegative when $x \geq 0$ and $t \geq 0$. The value of $x - ct$, however, is negative when $x < ct$. In this case the solution $u(x,t)$ is not represented by (9.2).

For the case $x - ct < 0$, the function $u(x,t)$ must still be the sum of a left-traveling wave and a right-traveling wave since $u(x,t)$ is a solution of the wave equation. The traveling wave represented by $G(x+ct)$ is still valid since $x + ct \geq 0$, but here we will replace

$F(x-ct)$ in (9.2) with a different traveling wave $F_1(x-ct)$ so that

$$u(x,t) = F_1(x-ct) + G(x+ct).$$

The case $x-ct < 0$ refers to the left-most part of the string $0 \leq x < ct$, and in particular includes the end of the string at $x = 0$. Since the string is pinned down at $x = 0$, the boundary condition $u(0,t) = 0$ requires

$$u(0,t) = F_1(-ct) + G(ct) = 0.$$

Letting $z = -ct$ shows that the function F_1 is defined by $F_1(z) = -G(-z)$ for all $z < 0$, and so the form of $u(x,t)$ is

$$u(x,t) = F_1(x-ct) + G(x+ct) = -G(ct-x) + G(x+ct).$$

Computing this expression with the function $G(x)$ from before, the value of $u(x,t)$ when $x - ct < 0$ is

$$u(x,t) = \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds. \tag{9.3}$$

Putting the two cases (9.2) and (9.3) together, the solution $u(x,t)$ of (9.1) is now piecewise defined. When $x \geq ct$, then the value of $u(x,t)$ is given by (9.2),

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \tag{9.4a}$$

while if $x > ct$, the value of $u(x,t)$ is given by (9.3),

$$u(x,t) = \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds. \tag{9.4b}$$

Example 9.1. The initial boundary value problem

$$\begin{aligned} u_{tt} &= 4u_{xx}, \quad 0 < x < \infty,\ t > 0, \\ u(x,0) &= e^{-(x-5)^2}, \\ u_t(x,0) &= 0, \\ u(0,t) &= 0 \end{aligned}$$

represents a semi-infinite string with its left end fixed. Traveling waves move along the string with speed $c = \sqrt{4} = 2$. Initially, the string is stretched along its equilibrium position, and then, at time $t = 0$, picked up around $x = 5$ and let go with no initial velocity.

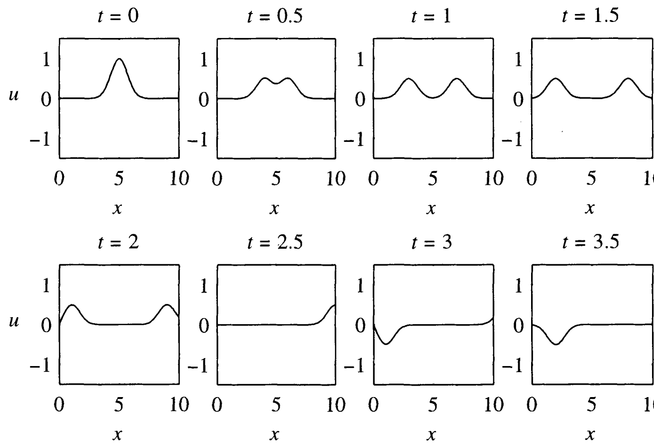


Figure 9.2. A left-traveling pulse encounters the fixed end of the string at $x = 0$ and reflects with opposing amplitude.

Substituting $c = 2$, $f(x) = e^{-(x-5)^2}$ and $g(x) = 0$ into (9.4) gives the solution

$$u(x,t) = \begin{cases} \frac{1}{2}e^{-(x+2t-5)^2} + \frac{1}{2}e^{-(x-2t-5)^2} & \text{if } x \geq 2t, \\ \frac{1}{2}e^{-(x+2t-5)^2} - \frac{1}{2}e^{-(2t-x-5)^2} & \text{if } x < 2t. \end{cases}$$

During the first couple seconds of animation, the solution resembles that of an infinite string. As shown in Figure 9.2, the initial shape $u(x,0) = e^{-(x-5)^2}$ splits into two smaller pulses, one traveling to the left and the other traveling to the right. In the following seconds, the right-traveling pulse continues to move to the right. The left-traveling pulse, however, encounters the fixed end of the string at $x = 0$. As the pulse reaches the fixed end, it reflects and becomes a pulse traveling to the right but with opposing amplitude.

Exercise 9.2. Consider the initial boundary value problem for a semi-infinite string

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < \infty,\ t > 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= xe^{-x^2}, \\ u(0,t) &= 0. \end{aligned}$$

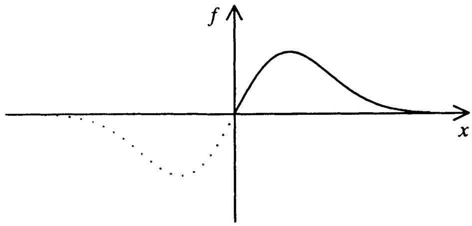


Figure 9.3. The odd extension of $f(x)$, $x \geq 0$.

The initial conditions of this problem represent a string which is initially at rest in its equilibrium position, then given a "kick" at time $t = 0$ which imposes an upward velocity of $u_t(x,0) = xe^{-x^2}$. Find the d'Alembert solution (9.4) of this semi-infinite problem. Note that here the piecewise function (9.4) simplifies to a single formula for all $x \geq 0$ and $t \geq 0$. Animate the solution.

Another approach for solving the initial boundary value problem (9.1) is to first extend the semi-infinite problem to a problem for the entire infinite line, and then use d'Alembert's formula. For example, consider the case of a semi-infinite string with zero initial velocity:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0, \\ u(0,t) &= 0. \end{aligned} \tag{9.5}$$

Let $f_o(x)$ be the **odd extension** of $f(x)$, defined by

$$f_o(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ -f(-x) & \text{if } x < 0, \end{cases}$$

and shown in Figure 9.3. A solution of the initial boundary value problem (9.5) can be constructed using the odd extension of the initial position and d'Alembert's formula (8.4) for the infinite string by

$$u(x,t) = \tfrac{1}{2}\left(f_o(x - ct) + f_o(x + ct)\right).$$

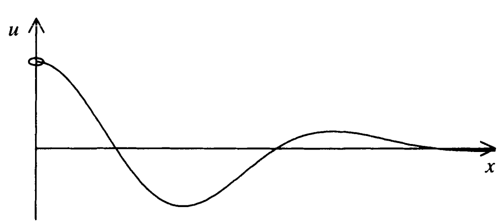
This is verified in the following exercise.

Exercise 9.3. Since $u(x,t) = \frac{1}{2}(f_o(x-ct) + f_o(x+ct))$ is the sum of traveling waves with speed c, $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$. Verify that $u(x,t)$ also satisfies the initial and boundary conditions in the initial boundary value problem (9.5).

Exercise 9.4. The companion MATLAB software (see page xiii) includes the script `wvstring` for animating the solution $u(x,t)$ of (9.1). In MATLAB, run this script and set $f(x) = e^{-(x-3)^2}$ and $g(x) = 0$. Animate the result with the x-range set to $[0, 10]$, then animate the result again with the x-range set to $[-10, 10]$ in order to see the effect of the odd extension of the initial data. The true part of the string is $x \geq 0$.

9.2. A semi-infinite string with free end

Suppose the left end of a semi-infinite string is attached to a ring which is allowed to move vertically along a frictionless vertical rod. The ring does not fix the left end, but instead it is assumed that the ring holds the end of the string "straight" out at $x = 0$:



This condition, written $u_x(0,t) = 0$, is a type of boundary condition commonly used to describe a free end of a string.

Following the steps used for the string with fixed end, the solution of the initial boundary value problem

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0, && \text{(PDE)}\\
u(x,0) &= f(x), && \text{(IC)}\\
u_t(x,0) &= g(x), && \\
u_x(0,t) &= 0 && \text{(BC)}
\end{aligned}
\tag{9.6}
$$

is also piecewise defined. As shown below in Exercise 9.5, if $x \geq ct$, then

$$(9.7a) \qquad u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

while if $x < ct$, then

$$(9.7b) \qquad u(x,t) = \frac{1}{2}\left(f(x+ct) + f(ct-x)\right) + \frac{1}{2c}\left[\int_{0}^{x+ct} g(s)ds + \int_{0}^{ct-x} g(s)ds\right].$$

Exercise 9.5. For the semi-infinite string with free end, let $u(x,t) = F_1(x-ct) + G(x+ct)$. Show that in order to satisfy the boundary condition $u_x(0,t) = 0$, F_1 must satisfy $F_1'(z) = -G'(-z)$, and so $F_1(z) = G(-z)$ for all $z = x - ct < 0$. Then follow the steps in Section 9.1 to derive (9.7).

Example 9.6. Consider Example 9.1, but with the left end free instead of fixed:

$$\begin{aligned} u_{tt} &= 4u_{xx}, \quad 0 < x < \infty,\ t > 0, \\ u(x,0) &= e^{-(x-5)^2}, \\ u_t(x,0) &= 0, \\ u_x(0,t) &= 0. \end{aligned}$$

Substituting $c = 2$, $f(x) = e^{-(x-5)^2}$ and $g(x) = 0$ into (9.7) gives the solution

$$u(x,t) = \begin{cases} \frac{1}{2}e^{-(x+2t-5)^2} + \frac{1}{2}e^{-(x-2t-5)^2} & \text{if } x \geq 2t, \\ \frac{1}{2}e^{-(x+2t-5)^2} + \frac{1}{2}e^{-(2t-x-5)^2} & \text{if } x < 2t. \end{cases}$$

Figure 9.4 shows several frames of animation of $u(x,t)$. Similar to the string with fixed end, the initial shape $u(x,0) = e^{-(x-5)^2}$ splits into two smaller pulses during the first couple seconds of animation, one traveling to the left and the other traveling to the right. In the following seconds, however, the left-traveling pulse reaches the free end $x = 0$ and reflects back to the right with the same amplitude and orientation.

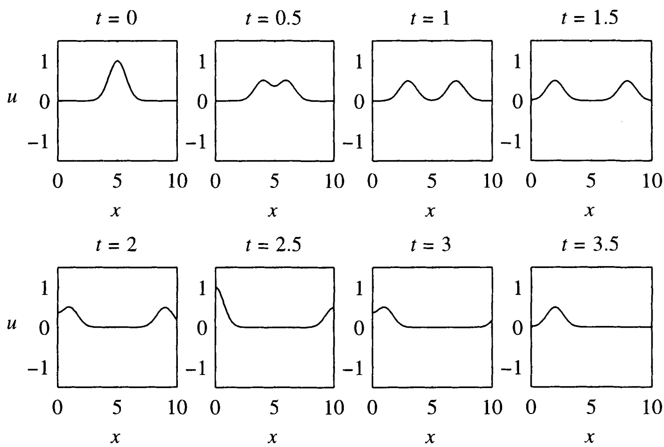


Figure 9.4. A left-traveling pulse encounters the free end and reflects with the same amplitude and orientation.

Exercise 9.7. The script `wvstring` provided with the companion MATLAB software (see page xiii) animates the d'Alembert solution of either (9.1) or (9.6). In MATLAB, run this script and animate the solution of Examples 9.1 and 9.6, comparing the behavior of the solution at the end of the string.

Chapter 10

Characteristic Lines of the Wave Equation

In the first chapter, a wave was described in terms of a disturbance, initially located in a small region of a medium, which then spreads as neighboring regions of the medium interact. The spreading of a disturbance through a medium will be examined in this chapter for waves governed by the wave equation. In particular, we will investigate how an initial disturbance described by the initial conditions $u(x,0)$ and $u_t(x,0)$ determines the solution to the wave equation in other parts of the medium at later times.

10.1. Domain of dependence and range of influence

In Section 8.2, it was shown that the solution of the initial value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x) \end{aligned}$$

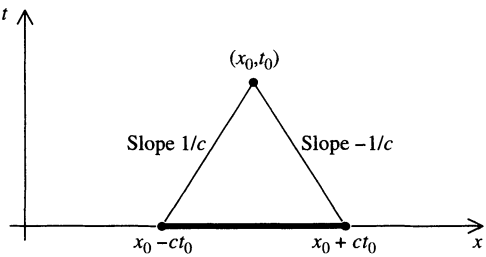


Figure 10.1. Domain of dependence $[x_0 - ct_0, x_0 + ct_0]$ of $u(x_0, t_0)$.

can be expressed using the d'Alembert formula as

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Since $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$, this can be rewritten as

(10.1)

$$u(x,t) = \frac{1}{2}\left(u(x-ct,0) + u(x+ct,0)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(s,0)ds.$$

This form emphasizes the fact that the value of u at a point (x_0, t_0) depends on the initial values of u and u_t, but only in the part of the medium between positions $x_0 - ct_0$ and $x_0 + ct_0$. The interval $[x_0-ct_0, x_0+ct_0]$ in the medium is called the **domain of dependence** for the solution u at (x_0, t_0).

Looking down on an xt–diagram, the domain of dependence refers to an interval $[x_0 - ct_0, x_0 + ct_0]$ on the x–axis, since it represents the points at time 0 which are used in the construction of the solution u at the point of later time (x_0, t_0). As shown in Figure 10.1, the domain of dependence for (x_0, t_0) can found by drawing lines of slope $1/c$ and $-1/c$ from (x_0, t_0) back to the x–axis. These lines are called **characteristic lines** or **characteristics** of the wave equation.

Alternatively, suppose that a disturbance is initially contained within an interval I in the medium. The **range of influence** of I is the collection of all points (x,t) in the xt–diagram whose domain of dependence includes some (or all) points of I (Figure 10.2). The

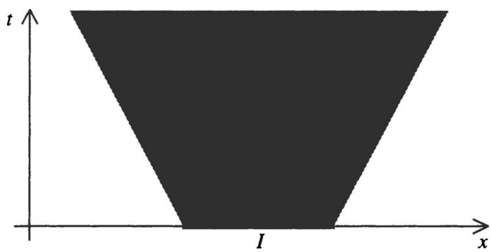


Figure 10.2. Range of influence of the interval I.

solution u at each point (x, t) in this set is "influenced" by some (or all) of the initial values of u and u_t along I on the $x-$axis. If (x, t) lies outside this set, then the value of u at position x and time t is not affected by the initial disturbance in I.

10.2. Characteristics and solutions of the wave equation

In the special case that $u_t(x, 0) = 0$ for all x, the d'Alembert solution (10.1) reduces to

$$u(x,t) = \frac{1}{2}\big(u(x - ct, 0) + u(x + ct, 0)\big).$$

This shows that the value of u at (x, t) depends only on the initial value of u at two points $x_1 = x - ct$ and $x_2 = x + ct$ in the medium. Once the initial values $u(x_1, 0)$ and $u(x_2, 0)$ are known, the solution u at (x, t) is constructed by taking the average of $u(x_1, 0)$ and $u(x_2, 0)$. In terms of an $xt-$diagram (see Figure 10.3), the value of $u(x, t)$ can be constructed by following the characteristic lines with slopes $\pm 1/c$ from (x, t) back to the $x-$axis and averaging the value of $u(x, 0)$ at the two resulting points $x_1 = x - ct$ and $x_2 = x + ct$ on the $x-$axis.

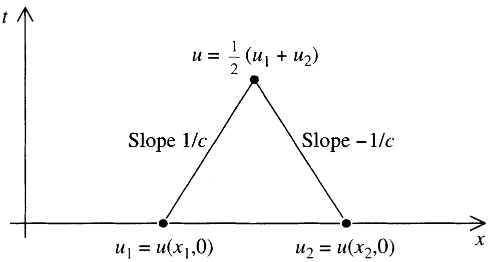


Figure 10.3. Value of u at (x, t) is the average of two initial values.

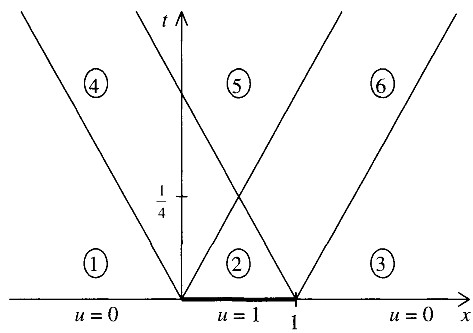


Figure 10.4. Six regions divided by four particular characteristic lines from Example 10.1.

Example 10.1. Characteristics and an $xt-$diagram will be used here to construct a solution of

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = 0.$$

The initial disturbance in this problem is contained within the interval $I = [0,1]$ of the medium, and the characteristic lines will have slopes $\pm 1/2$ since $c = 2$. Drawing left and right characteristic lines from the endpoints of I divides the $xt-$diagram into six regions (Figure 10.4); the range of influence of I consists of Regions 2, 4, 5, and 6.

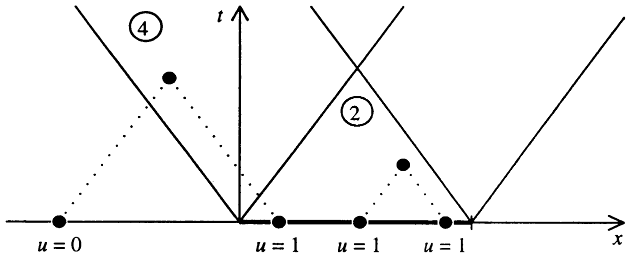


Figure 10.5. Extending characteristics back from (x, t) to the $x-$axis.

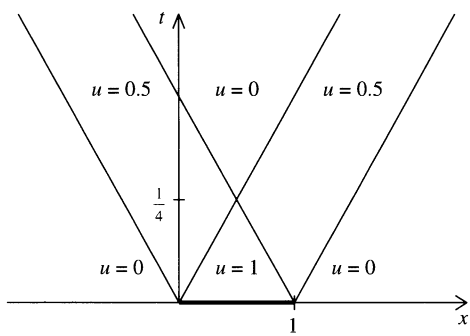


Figure 10.6. An $xt-$diagram of the solution u in Example 10.1.

As shown in Figure 10.5, each point (x, t) in Region 2 has the property that the two characteristic lines extending back from (x, t) to the $x-$axis will both end within the interval I. Since the value of u is 1 at each point in I, the value of u at (x, t) in Region 2 is the average $u = \frac{1}{2}(1 + 1) = 1$. At a point in Region 4, the characteristic line with positive slope extends back to the $x-$axis outside of I where the value of u is given to be 0 (Figure 10.5). At each point in Region 4, the value of u is then the average $u = \frac{1}{2}(0 + 1) = \frac{1}{2}$. Repeating this process for the remaining four regions constructs a function u which is constant within each of the six regions as shown in Figure 10.6.

The $xt-$diagram of the function $u(x, t)$ can be converted to frames of an animation by taking cross-sections in the $xt-$plane. Using the

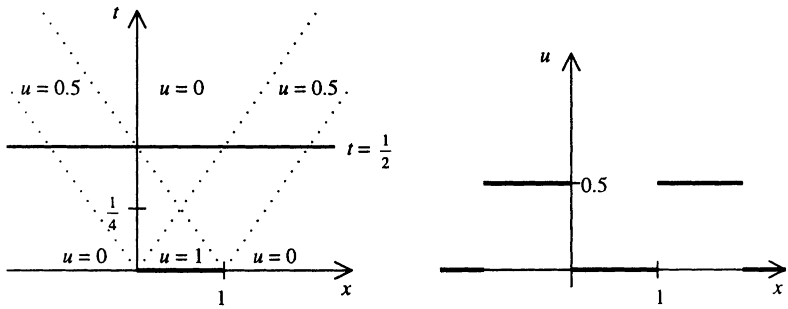


Figure 10.7. A slice of the xt–diagram at $t = 1/2$ produces the profile $u(x, 1/2)$.

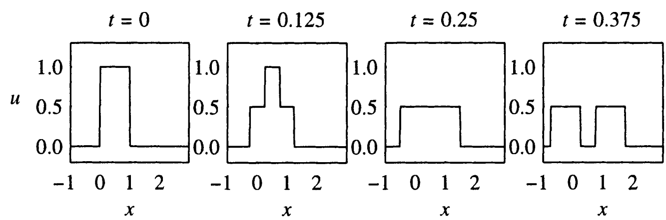


Figure 10.8. Four frames of animation of $u(x, t)$ from Example 10.1.

xt–diagram shown in Figure 10.6, the frame of animation showing the profile $u(x, 1/2)$ is constructed in Figure 10.7 by taking the horizontal slice of the xt–diagram at $t = 1/2$. Taking slices of Figure 10.6 at times $t = 0$, $1/8$, $1/4$, and $3/8$ results in the four frames of animation shown in Figure 10.8. The initial step shape $u(x, 0)$ splits into two smaller step-shaped pulses traveling to the right and to the left.

The function $u(x, t)$ constructed in the last example is not continuous and is not a solution of the wave equation $u_{tt} = c^2 u_{xx}$ along the characteristic lines dividing the six regions. The use of d'Alembert's formula, however, has constructed a function which is a solution of the wave equation in the interior of each region and which has properties similar to other solutions of the wave equation.

Exercise 10.2. Use characteristics to construct an xt–diagram representation (similar to Figure 10.6) for the d'Alembert solution of

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \; t > 0,$$
$$u(x,0) = \begin{cases} 0 & \text{if } 0 \le x \le 1, \\ 1 & \text{otherwise,} \end{cases}$$
$$u_t(x,0) = 0.$$

Exercise 10.3. Use characteristics to construct an xt–diagram representation for the d'Alembert solution of

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \; t > 0,$$
$$u(x,0) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ -1 & \text{if } 1 < x \le 2, \\ 0 & \text{otherwise,} \end{cases}$$
$$u_t(x,0) = 0.$$

Exercise 10.4. The solution of Example 10.1 can be written explicitly by writing the initial profile $u(x,0)$ in terms of the Heaviside function $H(x)$.

(a) Verify that $H(x)H(1-x)$ is a way of representing the given initial condition $u(x,0)$ in Example 10.1.

(b) Using $u(x,0) = H(x)H(1-x)$, write down the d'Alembert form of the solution of the initial value problem in Example 10.1 and animate the result. (The Heaviside function in Maple V is `Heaviside(x)`. If animating using MATLAB, the companion MATLAB software (page xiii) includes the function `heavi(x)` to compute $H(x)$).

(c) Plot the solution $u(x,t)$ from (b) as a surface. After changing the viewpoint so that you are looking directly down on the surface, the result should resemble the xt–diagram of the solution shown in Figure 10.6.

10.3. Solutions of the semi-infinite problem

Characteristics and an xt–diagram can also be used to construct solutions of the wave equation for a semi-infinite medium. Here we

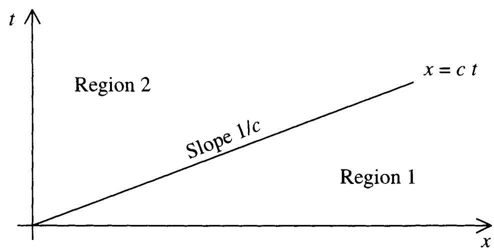


Figure 10.9

will examine the semi-infinite problem with zero initial velocity and left end pinned at zero:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty, \ t > 0, \\ u(x,0) &= f(x), \\ u_t(x,0) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

It was shown in Section 9.1 that the solution of this initial boundary value problem is piecewise defined and given by equations (9.4). When $u_t(x,0)$ is zero, $u(x,t)$ has the form

$$u(x,t) = \begin{cases} \frac{1}{2}\left(u(x+ct,0) + u(x-ct,0)\right) & \text{if } x \geq ct, \\ \frac{1}{2}\left(u(x+ct,0) - u(ct-x,0)\right) & \text{if } x < ct. \end{cases}$$

The line $x = ct$ which distinguishes between the two cases of this function also happens to be a characteristic of the wave equation. As shown in Figure 10.9, this line divides the first quadrant of the xt–plane into two regions.

In Region 1 of Figure 10.9 ($x > ct$), the solution is given by

$$u(x,t) = \tfrac{1}{2}\left(u(x+ct,0) + u(x-ct,0)\right),$$

and is the average of the initial value of u at positions $x + ct$ and $x - ct$ in the medium. As shown in Figure 10.10, this value can be constructed in the xt–diagram by drawing lines of slope $\pm 1/c$ from (x,t) back to the x–axis to identify the location of the points $x \pm ct$.

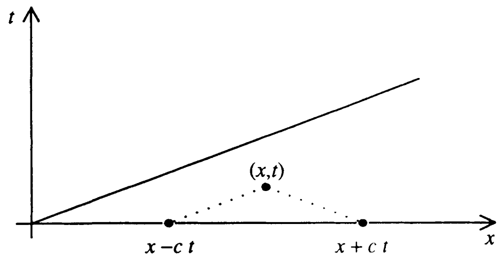


Figure 10.10. Value of u at (x, t) in Region 1 depends on the value of u at $(x \pm ct, 0)$.

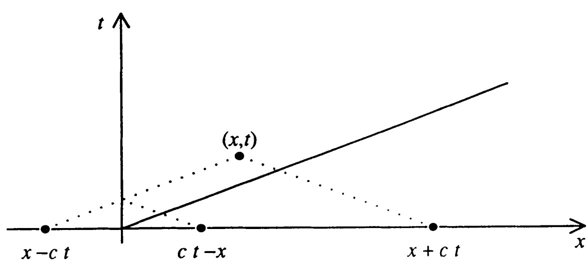


Figure 10.11. Value of u at (x, t) in Region 2 depends on the value of u at $(x + ct, 0)$ and $(ct - x, 0)$.

In Region 2 of Figure 10.9 ($x < ct$), the solution is given by

$$u(x,t) = \frac{1}{2}\left(u(x+ct,0) - u(ct-x,0)\right).$$

As shown in Figure 10.11, extending a line with slope $-1/c$ from (x, t) back down to the x–axis identifies the position $x + ct$. When extending the characteristic line with slope $1/c$ from (x, t), however, the characteristic encounters the t–axis. By reflecting off the t–axis back to the positive x–axis with slope $-1/c$, the resulting line meets the x–axis at the position $ct - x$. After locating positions $x_1 = ct - x$ and $x_2 = x + ct$ in the medium, the value of u at (x, t) is then found by computing $u(x, t) = \frac{1}{2}(u(x_2, 0) - u(x_1, 0))$.

Exercise 10.5. Use characteristics to construct an xt–diagram representation for the d'Alembert solution of

$$u_{tt} = u_{xx}, \quad 0 < x < \infty, \; t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = 0,$$

$$u(0,t) = 0.$$

Exercise 10.6. When the end of the string is free with boundary condition $u_x(0,t) = 0$, the d'Alembert form for $u(x,t)$ is given by equations (9.7) in Section 9.2. Taking $u_t(x,t) = 0$ in (9.7), how are characteristics used in the xt–diagram to construct the value of $u(x,t)$? Use characteristics to solve the previous problem with the fixed boundary condition $u(0,t) = 0$ replaced by the free boundary condition $u_x(0,t) = 0$.

Chapter 11

Standing Wave Solutions of the Wave Equation

The d'Alembert form for solutions of the wave equation was based on the observation that the general solution of $u_{tt} = c^2 u_{xx}$ could be decomposed into the sum of two traveling waves, each traveling with speed c but in opposite directions. Another approach for solving the wave equation involves decomposing the solution $u(x,t)$ into the sum of *standing waves.* In this chapter we will begin the discussion of this approach by computing standing wave solutions for the wave equation.

11.1. Standing waves

As shown previously in Section 8.1, the general solution of $u_{tt} = c^2 u_{xx}$ is $u(x,t) = F(x - ct) + G(x + ct)$, the sum of two traveling waves. The function

$$u(x,t) = \sin(x - t) + \sin(x + t)$$

is an example of such a solution, although the horizontal movement of a disturbance does not readily appear in its animation (Figure 11.1). The motion seen in Figure 11.1 is better explained by using a trigonometric identity to rewrite $u(x,t)$ as the product

$$u(x,t) = 2\cos(t)\sin(x).$$

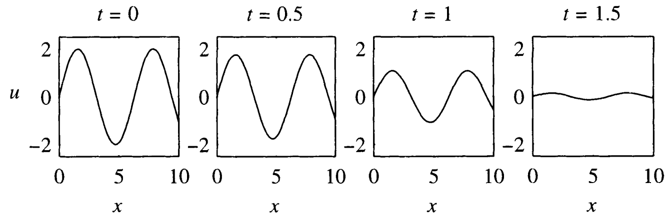


Figure 11.1. Profiles of $u(x,t) = \sin(x-t) + \sin(x+t)$.

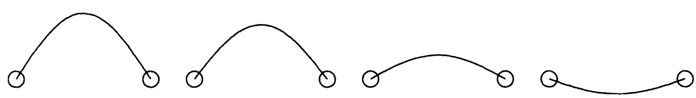


Figure 11.2. A simple motion of a string fixed at both ends.

In this form, the basic profile shape $v(x) = \sin(x)$ is being scaled by an amount $w(t) = 2\cos(t)$ at time t.

In general, a non-constant function of the form $u(x,t) = w(t)v(x)$ is called a **standing wave**. The animation of such a function shows the graph of the profile shape $v(x)$, which is then scaled vertically by an amount $w(t)$ at time t. When $w(t)$ is periodic in time, the result can resemble the motion of a vibrating string such as the one shown in Figure 11.2.

11.2. Standing wave solutions of the wave equation

Not every standing wave is a solution of the wave equation. In this section we will find those particular standing waves $u(x,t) = w(t)v(x)$ which are solutions of the wave equation in an infinite medium,

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0.$$

When $u(x,t) = w(t)v(x)$, the wave equation takes the form

$$w''(t)v(x) = c^2 w(t)v''(x).$$

Dividing both sides by $w(t)v(x)$ gives the equation

$$\frac{w''(t)}{w(t)} = c^2 \frac{v''(x)}{v(x)}$$

where the left side depends only on t and the right side depends only on x. Since the left side is independent of x, the right side must also be independent of x, and so the right side must be constant. We can then write

$$\frac{w''(t)}{w(t)} = c^2 \frac{v''(x)}{v(x)} = \lambda$$

for some constant λ, resulting in the two ordinary differential equations

$$w''(t) = \lambda w(t), \quad v''(x) = \frac{\lambda}{c^2} v(x). \tag{11.1}$$

The solutions of these equations depend upon whether λ is zero, positive, or negative.

If $\lambda = 0$, then the two differential equations in (11.1) reduce to

$$w''(t) = 0, \quad v''(x) = 0.$$

In this case $w(t) = A + Bt$ and $v(x) = C + Dx$ for arbitrary constants A, B, C, and D, resulting in the standing waves

$$u(x,t) = (A + Bt)(C + Dx). \tag{11.2}$$

If $\lambda > 0$, then λ can be rewritten as $\lambda = r^2$ for some $r > 0$. In this case the differential equations in (11.1) are of the form

$$w''(t) = r^2 w(t), \quad v''(x) = \left(\frac{r}{c}\right)^2 v(x).$$

The general solutions here are $w(t) = Ae^{rt} + Be^{-rt}$ and $v(x) = Ce^{rx/c} + De^{-rx/c}$, giving the standing waves

$$u(x,t) = (Ae^{rt} + Be^{-rt})(Ce^{rx/c} + De^{-rx/c}). \tag{11.3}$$

If $\lambda < 0$, then λ can be rewritten as $\lambda = -r^2$ for some $r > 0$. In this case the differential equations in (11.1) are of the form

$$w''(t) = -r^2 w(t), \quad v''(x) = -\left(\frac{r}{c}\right)^2 v(x).$$

The general solutions here are $w(t) = A\cos(rt) + B\sin(rt)$ and $v(x) = C\cos(rx/c) + D\sin(rx/c)$, resulting in the standing waves

$$u(x,t) = (A\cos(rt) + B\sin(rt))(C\cos(rx/c) + D\sin(rx/c)). \tag{11.4}$$

Functions $u(x,t)$ of the form (11.2), (11.3), and (11.4) describe all possible standing waves which are solutions of the wave equation. In subsequent sections we will select out those which meet certain physical conditions.

Exercise 11.1. For the wave equation $u_{tt} = 9u_{xx}$, construct an example of a standing wave solution for each of the cases $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$ by selecting values for A, B, C, D, and r. Animate the results.

11.3. Standing waves of a finite string

The previous section gathered together all possible standing wave solutions of the wave equation. In particular applications, however, only a small subset of these may be physically realistic.

As a particular application, we will find the standing waves for a string of finite length L in which both ends of the string are fixed,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= 0, \\ u(L,t) &= 0. \end{aligned} \tag{11.5}$$

If a standing wave $u(x,t) = w(t)v(x)$ is to satisfy the left boundary condition $u(0,t) = v(0)w(t) = 0$ for all t, then either $v(0) = 0$ or $w(t) = 0$ for all t. The possibility $w(t) = 0$ results in the zero function $u(x,t) = w(t)v(x) = 0$, which is not considered a standing wave, so $v(0) = 0$ is required of the function $v(x)$. Similarly, the right boundary condition $u(L,t) = v(L)w(t) = 0$ will require $v(x)$ to satisfy $v(L) = 0$. Thus, in order for $u(x,t) = w(t)v(x)$ to be a standing wave solution of (11.5), $v(x)$ must be a function such that

$$v(0) = 0, \quad v(L) = 0. \tag{11.6}$$

In the previous section it was found that standing wave solutions of the wave equation $u_{tt} = c^2 u_{xx}$ have the three basic profile shapes

$$v(x) = C + Dx,$$
$$v(x) = Ce^{rx/c} + De^{-rx/c},$$
$$v(x) = C\cos(rx/c) + D\sin(rx/c).$$

The task now is to determine which of these profile shapes can match the boundary conditions (11.6).

In order for the linear form $v(x) = C+Dx$ to satisfy the boundary conditions $v(0) = 0$ and $v(L) = 0$ in (11.6), C and D must satisfy

$$C = 0,$$
$$C + DL = 0.$$

In this case C and D must both be 0, so $v(x) = 0$ for all x. This, however, would result in $u(x,t) = w(t)v(x) = 0$, which is not considered a standing wave. There are no standing wave solutions for the fixed string with a linear profile shape.

In order for the exponential form $v(x) = Ce^{rx/c} + De^{-rx/c}$ to satisfy (11.6), C and D must satisfy the two equations

$$C + D = 0,$$
$$Ce^{rL/c} + De^{-rL/c} = 0.$$

Multiplying the first equation by $-e^{-rL/c}$ and adding the result to the second equation shows that $C(e^{rL/c} - e^{-rL/c}) = 0$. Since $rL/c \neq -rL/c$, then $e^{rL/c} \neq e^{-rL/c}$, and so the only choice for C is 0. From $C + D = 0$ the value of D must also be zero, and so $v(x) = 0$ for all x. This profile shape, however, results in $u(x,t) = w(t)v(x) = 0$ and is not considered a standing wave.

The third possible profile shape is the trigonometric form $v(x) = C\cos(rx/c)+D\sin(rx/c)$. In order for this shape to satisfy the boundary conditions (11.6), C and D must be satisfy the two equations

$$C = 0,$$
$$D\sin(rL/c) = 0.$$

The second equation implies either $D = 0$ or $\sin(rL/c) = 0$; however, since $D = 0$ would result in $v(x) = 0$ (no standing wave), the only remaining option is $\sin(rL/c) = 0$. In this case rL/c must be a multiple of π, that is, $rL/c = n\pi$ for some integer n. The constants L and c are physical constants which describe the string, and so the constant r must then be picked to be

$$r = \frac{n\pi c}{L}.$$

For every (nonzero) integer n we get a standing wave solution of the form (11.4) with $r = n\pi c/L$:

$$u(x,t) = [A\cos(n\pi ct/L) + B\sin(n\pi ct/L)]D\sin(n\pi x/L).$$

By renaming the arbitrary constants AD and BD as A and B, these standing waves will be written as

$$\boxed{u_n(x,t) = [A\cos(n\pi ct/L) + B\sin(n\pi ct/L)]\sin(n\pi x/L).} \tag{11.7}$$

These are the only standing waves for a vibrating string with fixed ends.

Exercise 11.2. If a string with length L has its left end fixed but the right end is free, then the boundary condition at $x = L$ changes to $u_x(L,t) = 0$. Find the standing wave solutions of

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= 0, \\ u_x(L,t) &= 0. \end{aligned}$$

Exercise 11.3. At this point we have omitted initial conditions for the string. By picking appropriate choices of the constants n, A, and B in (11.7), find the particular standing wave solutions of

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0 \end{aligned}$$

which satisfy the following initial conditions:

(a) $u(x,0) = 10\sin(\pi x)$ and $u_t(x,0) = 0$.

(b) $u(x,0) = 0$ and $u_t(x,0) = -3\sin(2\pi x)$.

(c) $u(x,0) = \sin(4\pi x)$ and $u_t(x,0) = 2\sin(4\pi x)$.

Animate each solution.

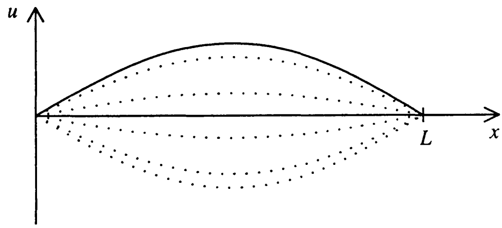


Figure 11.3. First or Fundamental Mode ($n = 1$)

11.4. Modes of vibration

The standing wave solution (11.7)

$$u_n(x,t) = [A\cos(n\pi ct/L) + B\sin(n\pi ct/L)]\sin(n\pi x/L)$$

of the wave equation for a string with fixed ends is a relatively simple type of motion of the string called the n^{th} **mode of vibration**. By using a trigonometric identity, the time varying part

$$A\cos(n\pi ct/L) + B\sin(n\pi ct/L)$$

can be rewritten as

$$R\cos(n\pi ct/L - \delta)$$

where R and δ are constants in terms of A and B. The standing wave then becomes

$$u_n(x,t) = R\cos(n\pi ct/L - \delta)\sin(n\pi x/L) \tag{11.8}$$

with a profile shape $\sin(n\pi x/L)$ whose amplitude is scaled periodically between $-R$ and R. The first three modes of vibration are shown in Figures 11.3–11.5.

The form of the standing wave (11.8) also shows that higher modes of vibration "beat" at higher frequencies, which we then hear as higher tones. From $w(t) = R\cos(n\pi ct/L - \delta)$ it follows that the n^{th} mode of vibration completes $n\pi c/L$ oscillations between $-R$ and R in 2π seconds. The number

$$\omega_n = \frac{n\pi c}{L}$$

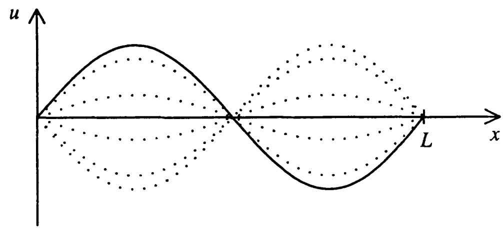


Figure 11.4. Second Mode ($n = 2$)

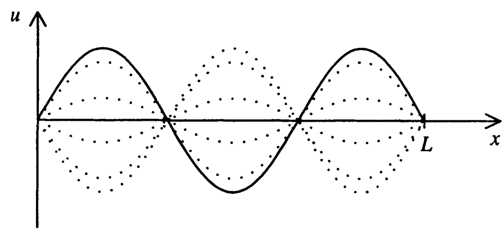


Figure 11.5. Third Mode ($n = 3$)

is called the **circular frequency** of the n^{th} mode. In practice it is customary to refer to the number of complete oscillations in one second, computed by

$$f_n = \frac{n\pi c/L}{2\pi} = \frac{nc}{2L} \quad \text{cycles per second (Hertz).}$$

The number f_n is called the **frequency** of the mode, while the numbers $\{f_1, f_2, f_3, \dots\}$ are called the **natural frequencies** of the string. The first natural frequency f_1 is often called the fundamental tone, while the higher frequencies are called overtones.

Exercise 11.4. Find the standing wave solutions for a finite string which is free at both ends:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u_x(0,t) &= 0, \\ u_x(L,t) &= 0. \end{aligned}$$

What does the animation of these modes of vibration look like?

Exercise 11.5. Recall from the derivation of the wave equation $u_{tt} = c^2 u_{xx}$ in Section 7.2 that $c = \sqrt{T/\rho}$ where T is the tension of the string and ρ is the constant density. This implies that the natural frequencies of the string with fixed ends are given by

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\rho}}.$$

(a) Explain what happens to the sound (frequencies) of the string as the tension is increased. What happens when the density of the string is increased?

(b) Steel guitar strings have densities which typically range from 0.40 to 2.0 grams per meter. If a guitar string 0.6 meters long with density 0.40 grams per meter is tightened to a tension of 15 Newton-meters, what are the predicted natural frequencies (in Hertz) of the string?

When looking for standing wave solutions $u(x,t) = w(t)v(x)$ for the string with two fixed ends, the critical part was finding profile shapes $v(x)$ which satisfied the differential equation for $v(x)$ in (11.1) and the boundary conditions (11.6). The related boundary value problem

$$\begin{aligned} -c^2 v'' &= \lambda v, \quad \text{for } 0 < x < L\ , \\ v(0) &= 0, \\ v(L) &= 0 \end{aligned}$$

is called a *Sturm-Liouville* problem. An extra minus sign has been added in the differential equation—the λ here is opposite in sign from the value of λ used in Section 11.2.

The values $\lambda_n = (n\pi c/L)^2$ which yield a nonzero solution $v(x)$ of the Sturm-Liouville problem are called *eigenvalues.* Note that the circular frequency of vibration of the n^{th} mode of vibration is in fact the square root of an eigenvalue, $\omega_n = n\pi c/L = \sqrt{\lambda_n}$. The corresponding solutions $v_n(x) = D\sin(n\pi x/L)$ are called *eigenfunctions.*

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Chapter 12

Standing Waves of a Nonhomogeneous String

12.1. The wave equation for a nonhomogeneous string

The wave equation $u_{tt} = c^2 u_{xx}$ was derived as a description of a string with constant density ρ and tension T where $c^2 = T/\rho$. When the density or tension varies from position to position along a string, the string is called **nonhomogeneous**. Returning to the derivation of the wave equation in Section 7.2, recall the approximation (7.4) to Newton's second law of motion for a portion of the string lying above the interval $[x, x + \Delta x]$ on the x–axis. If T and ρ depend on x, then this approximation becomes

$$\rho(x)u_{tt}(x,t) = T(x + \Delta x)u_x(x + \Delta x, t) - T(x)u_x(x,t).$$

Dividing by Δx and letting $\Delta x \to 0$ gives a more general form of the wave equation,

$$\boxed{\rho(x)u_{tt}(x,t) = \left(T(x)u_x(x,t)\right)_x .} \tag{12.1}$$

In the rest of this chapter we will find the standing wave solutions of this equation and examine the modes of vibration for a nonhomogeneous string.

12.2. Standing waves of a finite string

The process of looking for standing wave solutions of the more general wave equation (12.1) is similar to the one used for the constant coefficient wave equation $u_{tt} = c^2 u_{xx}$. As in Section 11.3, consider a string of finite length L in which both ends are held fixed. If the string is nonhomogeneous, then the boundary value problem for the displacement $u(x,t)$ of the string is

$$\begin{aligned} \rho(x)u_{tt}(x,t) &= (T(x)u_x(x,t))_x, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= 0, \\ u(L,t) &= 0. \end{aligned}$$

To find the standing wave solutions $u(x,t) = w(t)v(x)$ of this problem, substitute $u(x,t) = w(t)v(x)$ into the wave equation to get

$$\rho(x)v(x)w''(t) = (T(x)u'(x))'w(t),$$

and then divide both sides by $v(x)w(t)$ to separate the variables into

$$\frac{w''(t)}{w(t)} = \frac{(T(x)v'(x))'}{\rho(x)v(x)}.$$

Since the left side does not depend on x, the right side is also independent of x and so is constant. Letting this constant be $-\lambda$ and writing

$$\frac{w''(t)}{w(t)} = \frac{(T(x)v'(x))'}{\rho(x)v(x)} = -\lambda$$

produces the two ordinary differential equations

$$(12.2) \qquad -w''(t) = \lambda w(t), \quad -(T(x)v'(x))' = \lambda\rho(x)v(x).$$

The extra minus sign in the constant $-\lambda$ is not necessary, but is chosen for later convenience.

Finding standing wave solutions $u(x,t) = w(t)v(x)$ requires finding solutions of the two ordinary differential equations (12.2). The differential equation for $w(t)$ has solutions which depend on whether

λ is positive, negative, or zero:

$$\begin{aligned} w(t) &= A + Bt && \text{if } \lambda = 0, \\ w(t) &= Ae^{-\sqrt{-\lambda}t} + Be^{\sqrt{-\lambda}t} && \text{if } \lambda < 0, \\ w(t) &= A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t) && \text{if } \lambda > 0. \end{aligned} \tag{12.3}$$

The differential equation for $v(x)$, however, cannot be solved explicitly in general, and so we will simply let $v(x)$ denote a solution of $-(T(x)v'(x))' = \lambda\rho(x)v(x)$. If $u(x,t) = w(t)v(x)$ is nonconstant, then $u(x,t)$ forms a standing wave solution of the wave equation (12.1).

The function $u(x,t) = w(t)v(x)$ is a standing wave of a finite string with fixed ends $u(0,t) = u(L,t) = 0$ if the profile shape $v(x)$ is a nonzero solution of

$$\begin{aligned} -(T(x)v'(x))' &= \lambda\rho(x)v(x), \quad 0 < x < L, \\ v(0) &= 0, \\ v(L) &= 0. \end{aligned} \tag{12.4}$$

The function $v(x) = 0$ for all x certainly satisfies this boundary value problem regardless of the choice of λ; however, $u(x,t) = w(t)v(x) = 0$ is not considered a standing wave. It can be shown that if $\rho(x)$ is a positive continuous function and $T(x)$ is a positive differentiable function, then the only values of λ which permit nonzero solutions of the boundary value problem (12.4) are positive and form an increasing sequence of real numbers

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots .$$

The proof of this can be found in a number of advanced books on ordinary differential equations and boundary value problems; see [**MM**, Chapter 4], for example. With such a λ_n in the boundary value problem (12.4) and its resulting nonzero solution $v_n(x)$, the standing wave $u(x,t) = w(t)v(x)$ is completed by finding $w(t)$. Since $\lambda = \lambda_n > 0$, the function $w(t)$ is constructed by the third form in (12.3). The resulting function $u(x,t) = w(t)v(x)$ given by

$$\boxed{u_n(x,t) = (A\cos(\sqrt{\lambda_n}t) + B\sin(\sqrt{\lambda_n}t))v_n(x)} \tag{12.5}$$

forms a standing wave solution of the nonhomogeneous string with fixed ends.

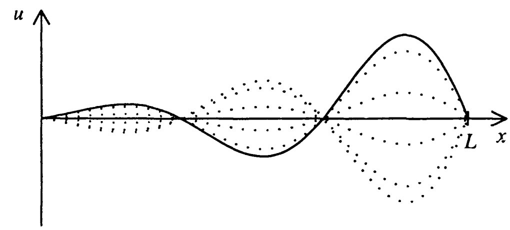

Figure 12.1. The third mode of vibration for a nonuniform string.

12.3. Modes of vibration

The standing wave solution (12.5) is called the n^{th} **mode of vibration** of the string with fixed ends. As in the case of constant density ρ and tension T, this mode of vibration can be rewritten as

$$u_n(x,t) = R\cos(\sqrt{\lambda_n}t - \delta)v_n(x)$$

for constants R and δ found in terms of A and B. This shows that the standing wave has a profile shape of $v_n(x)$ that is scaled vertically by an amount which oscillates between $-R$ and R. The (circular) frequency at which the profile shape oscillates is $\omega_n = \sqrt{\lambda_n}$ cycles per 2π seconds, or

$$f_n = \frac{\sqrt{\lambda_n}}{2\pi}$$

cycles per second (Hertz). Each f_n is called a **natural frequency** of the string.

In Section 11.3 it was shown that if $T(x)$ and $\rho(x)$ are constant (with $c^2 = T/\rho$), then the profile shape of the n^{th} mode of vibration is $\sin(n\pi x/L)$. When $T(x)$ and $\rho(x)$ are not constant it is generally not possible to explicitly solve (12.4) to determine the profile shape $v_n(x)$ of the n^{th} mode of vibration. The mode shape, however, will closely resemble $\sin(n\pi x/L)$. An example of the third mode of vibration for a nonuniform string is shown in Figure 12.1; compare this with the third mode of a uniform string shown in Figure 11.5. The restricted vibration of the string in this figure suggests that the left part of the string is denser and/or under higher tension.

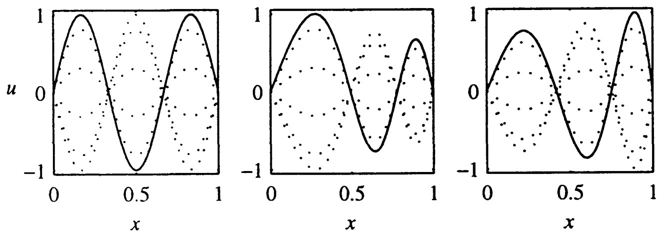

Figure 12.2. Third modes of vibration for three different strings in Exercise 12.1.

Exercise 12.1. Shown in Figure 12.2 are the third modes of vibration for three different strings. Use your intuition about string density and tension to match the following string densities and tensions with the mode of vibration.

(a) $\rho(x) = 1 + 10x^2$, $T(x) = 1$, (b) $\rho(x) = 1$, $T(x) = 5 - 4x$, (c) $\rho(x) = 1$, $T(x) = 1$.

12.4. Numerical calculation of natural frequencies

In Section 11.4 it was shown that if $T(x)$ and $\rho(x)$ are constant, then the natural frequencies of the string are

$$\frac{c}{2L}, \quad \frac{2c}{2L}, \quad \frac{3c}{2L}, \quad \ldots$$

where $c = \sqrt{T/\rho}$. When $T(x)$ and $\rho(x)$ are not constant, the n^{th} frequency is given by $f_n = \sqrt{\lambda_n}/(2\pi)$ where λ_n is the n^{th} value of λ for which the boundary value problem (12.4) has a nonzero solution $v_n(x)$. In general, however, it is not possible to explicitly compute the values of λ_n. One procedure for constructing numerical approximations of these values of λ is called the *shooting method.*

To illustrate this method, we will look for the first natural frequency $f_1 = \sqrt{\lambda_1}/(2\pi)$ of a string by finding an approximation to λ_1. The profile $v_1(x)$ of this mode shape will resemble the shape $\sin(\pi x/L)$ of a uniform string. In particular, $v_1(x)$ will satisfy $v(0) = 0$ and

$v(L) = 0$, and will have only one local extreme point between $x = 0$ and $x = L$.

An initial approximation of λ_1 can be made by approximating the string tension $T(x)$ and density $\rho(x)$ by their *average* values,

$$T_0 = \frac{1}{L}\int_0^L T(x)dx, \quad \rho_0 = \frac{1}{L}\int_0^L \rho(x)dx.$$

Replacing $T(x)$ and $\rho(x)$ with T_0 and ρ_0 in (12.4) yields the boundary value problem

$$\begin{aligned} -v''(x) &= c^2 v(x), \quad 0 < x < L, \\ v(0) &= 0, \\ v(L) &= 0 \end{aligned}$$

where $c = \sqrt{T_0/\rho_0}$. The first natural frequency of the string represented by this constant coefficient problem is $f_1^{(0)} = c/(2L)$, so an initial approximation of λ_1 is $\lambda_1^{(0)} = (2\pi f_1^{(0)})^2 = \pi^2 T_0/(\rho_0 L^2)$.

An indication of how close the approximation $\lambda_1^{(0)}$ is to λ_1 can be made by solving the following *initial* value problem:

$$\begin{aligned} -(T(x)v'(x))' &= \lambda_1^{(0)}\rho(x)v(x), \quad 0 < x < L, \\ v(0) &= 0, \\ v'(0) &= 1. \end{aligned} \tag{12.6}$$

In general this cannot be done explicitly; however, most computer algebra systems or numerical software packages contain routines for numerically solving initial value problems such as this one. Assuming the solution $v(x)$ of this initial value problem has been computed, examine the value of $v(x)$ at the right endpoint $x = L$. If $v(L)$ is zero, then $v(x)$ satisfies all of the conditions of the boundary value problem (12.4), namely

$$\begin{aligned} -(T(x)v'(x))' &= \lambda_1^{(0)}\rho(x)v(x), \quad 0 < x < L, \\ v(0) &= 0, \\ v(L) &= 0. \end{aligned}$$

In this case, if the solution $v(x)$ resembles $\sin(\pi x/L)$, then $v(x)$ is the profile shape of the first mode of vibration, and $\lambda_1^{(0)}$ is exactly λ_1.

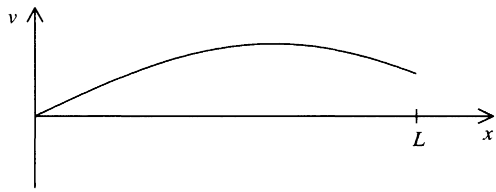

Figure 12.3. Solution of the initial value problem (12.6) when $\lambda_1^{(0)} < \lambda_1$.

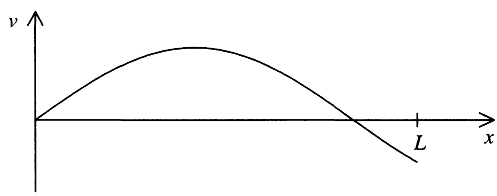

Figure 12.4. Solution of the initial value problem (12.6) when $\lambda_1^{(0)} > \lambda_1$.

In practice, the value of $v(L)$ will not be zero after solving the initial value problem (12.6), since $\lambda_1^{(0)}$ was only an approximation of λ_1. If $v(L) > 0$ (see Figure 12.3), then the value of $\lambda_1^{(0)}$ was too small and the next guess for λ_1 should be made larger. If $v(L) < 0$ (see Figure 12.4), then the value of $\lambda_1^{(0)}$ was too large and the next guess for λ_1 should be smaller.

A bisection approach can be used to create an iteration scheme for finding the next approximation of λ_1: if $v(L) > 0$ when $\lambda = a$ and $v(L) < 0$ when $\lambda = b$, then let the next approximation be the midpoint value $\lambda_1^{(1)} = \frac{1}{2}(a+b)$. As this process is repeated, the value of $v(L)$ becomes closer to 0 as the value of $\lambda_1^{(k)}$ approaches λ_1. The fundamental frequency of the string is then estimated at each step by $f_1 \approx \sqrt{\lambda_1^{(k)}}/(2\pi)$.

Exercise 12.2. Consider a string with fixed ends and length $L = 1$ which has constant tension $T = 1$ but variable density $\rho(x) = 1 + x^2$. The modes of vibration for this string have profile shapes $v(x)$ which are solutions of

$$-v''(x) = \lambda_n(1 + x^2)v(x), \quad 0 < x < 1,$$
$$v(0) = 0,$$
$$v(1) = 0.$$

(a) Compute the average tension T_0 and average density ρ_0 of the string. Then let $c = \sqrt{T_0/\rho_0}$ and form an initial approximation $f_1^{(0)} = c/(2L)$ of the string's first frequency f_1.

(b) Compute $\lambda_1^{(0)} = (2\pi f_1^{(0)})^2$. Then use computer software or a graphing calculator to numerically solve

$$-v''(x) = \lambda_1^{(0)}(1 + x^2)v(x), \quad 0 < x < 1,$$
$$v(0) = 0,$$
$$v'(0) = 1.$$

A graph of the result $v(x)$ should resemble $\sin(\pi x)$ over the interval $[0, 1]$; however $v(1)$ is not zero.

(c) Based on part (b), is the value $\lambda_1^{(0)}$ smaller or larger than λ_1? Use this to make a new guess $\lambda_1^{(1)}$ for λ_1 and repeat part (b) with $\lambda_1^{(0)}$ replaced by $\lambda_1^{(1)}$. If needed, choose different values of $\lambda_1^{(1)}$ and repeat part (b) until the value of $v(1)$ is approximately zero.

(d) Once you have found a value of $\lambda_1^{(1)}$ for which $v(1)$ is close to zero, estimate the first natural frequency of the string by $f_1 \approx \sqrt{\lambda_1^{(1)}}/(2\pi)$.

(e) The resulting graph of $v(x)$ made by solving part (b) with $\lambda_1^{(1)}$ will be the profile shape of the first mode of vibration. Does the profile shape appear to be biased towards the heavier end of the string or the lighter end of the string?

Chapter 13

Superposition of Standing Waves

The sum of two traveling waves can form a standing wave, such as in the example $u(x,t) = \sin(x-t) + \sin(x+t) = 2\cos(t)\sin(x)$. By adding standing waves together, we can form more general functions called *compound waves.* Just as the d'Alembert formula used traveling waves as the building blocks from which to construct solutions of the wave equation for an infinite string, here we will use standing waves as the basic building blocks for constructing solutions of the wave equation for a finite string.

13.1. Finite superposition

If $u(x,t)$ and $v(x,t)$ are functions which represent two waves, then the function $w(x,t) = au(x,t) + bv(x,t)$ is called a **superposition** of u and v. The **principle of superposition** is a property of linear homogeneous differential equations which allows one to construct new solutions by superimposing two or more known solutions together.

First we will verify that the boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= 0, \\ u(L,t) &= 0 \end{aligned} \tag{13.1}$$

has the property that a superposition of two of its solutions is also a solution. If $u(x,t)$ and $v(x,t)$ are two solutions of (13.1), then

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & v_{tt} &= c^2 v_{xx}, \\ u(0,t) &= 0, & v(0,t) &= 0, \\ u(L,t) &= 0, & v(L,t) &= 0. \end{aligned}$$

Let $w(x,t)$ be the superposition constructed by forming the combination $w = au + bv$ for some constants a and b. Then

$$\begin{aligned} w_{tt} &= (au + bv)_{tt} \\ &= au_{tt} + bv_{tt} \\ &= a(c^2 u_{xx}) + b(c^2 v_{xx}) \\ &= c^2 (au + bv)_{xx} \\ &= c^2 w_{xx} \end{aligned}$$

shows that w is also a solution of the wave equation. The values of $w(x,t)$ at the boundaries $x = 0$ and $x = L$ are

$$\begin{aligned} w(0,t) &= au(0,t) + bv(0,t) = a \cdot 0 + b \cdot 0 = 0, \\ w(L,t) &= au(L,t) + bv(L,t) = a \cdot 0 + b \cdot 0 = 0, \end{aligned}$$

so w satisfies the same boundary conditions as u and v. This shows that the superposition w of solutions of the boundary value problem (13.1) is yet another solution of (13.1).

Exercise 13.1. Show that the principle of superposition does not apply to solutions of the following boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < L,\ t > 0, \\ u(0,t) &= 1, \\ u(L,t) &= 0. \end{aligned}$$

In Section 11.3, the standing waves

$$u_n(x,t) = [A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)] \sin(n\pi x/L)$$

were found to be solutions of the boundary value problem (13.1) for each positive integer n. By the principle of superposition, the sum

$$u(x,t) = u_1(x,t) + u_2(x,t) + \cdots + u_N(x,t) = \sum_{n=1}^{N} u_n(x,t)$$

creates yet another solution of (13.1), given by

$$u(x,t) = \sum_{n=1}^{N} [A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)] \sin(n\pi x/L). \tag{13.2}$$

The superposition of standing waves such as this is called a **compound wave**. The constants A_n and B_n are arbitrary and can be picked so that $u(x,t)$ represents a number of different initial conditions.

Example 13.2. Consider the initial boundary value problem for a string with length $L = 1$ and wave speed $c = 1$:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= 2\sin(\pi x) - 3\sin(2\pi x). \end{aligned}$$

By superimposing standing wave solutions, the compound wave (13.2) given by

$$u(x,t) = \sum_{n=1}^{N} [A_n \cos(n\pi t) + B_n \sin(n\pi t)] \sin(n\pi x) \tag{13.3}$$

also satisfies the wave equation and the two boundary conditions. At $t = 0$ the initial position

$$u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x)$$

is supposed to be zero; this can be accomplished by selecting each A_n to be 0. The velocity

$$u_t(x,t) = \sum_{n=1}^{N} [-A_n n\pi \sin(n\pi t) + B_n n\pi \cos(n\pi t)] \sin(n\pi x)$$

at $t = 0$ is

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{N} n\pi B_n \sin(n\pi x) \\ &= \pi B_1 \sin(\pi x) + 2\pi B_2 \sin(2\pi x) + \cdots + N\pi B_N \sin(N\pi x). \end{aligned}$$

This matches the given initial velocity $2\sin(\pi x) - 3\sin(2\pi x)$ if we select $N = 2$ and pick the coefficients B_n so that

$$\pi B_1 = 2, \quad 2\pi B_2 = -3.$$

With $N = 2$, $B_1 = 2\pi$, $B_2 = -3/(2\pi)$ and $A_1 = A_2 = 0$, the solution (13.3) to the given initial boundary value problem is then

$$u(x,t) = \frac{2}{\pi}\sin(\pi t)\sin(\pi x) - \frac{3}{2\pi}\sin(2\pi t)\sin(2\pi x).$$

Exercise 13.3. Find the solutions of

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0 \end{aligned}$$

which satisfy the following initial conditions:

(a) $u(x,0) = 10\sin(\pi x) + 3\sin(4\pi x)$ and $u_t(x,0) = 0$.

(b) $u(x,0) = \sin(2\pi x)$ and $u_t(x,0) = -3\sin(2\pi x)$.

Animate the results, noting that these solutions are not standing waves.

13.2. Infinite superposition

In the previous section, the principle of superposition was used to combine a finite number of standing waves to form a more general compound wave. Taking superposition one step further, one can also consider the superposition of all of the standing wave solutions $u_n(x,t)$ of (13.1) to get

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(n\pi ct/L) + B_n\sin(n\pi ct/L)]\sin(n\pi x/L). \tag{13.4}$$

Some care must be taken, since this is an infinite series and not all infinite series converge. If the numbers A_n and B_n, however, decrease to zero quite rapidly as $n \to \infty$, then the series is guaranteed to converge to a function $u(x,t)$ which is still a solution of the boundary value problem (13.1). Armed with this general solution, our task again is to pick the coefficients A_n and B_n so that $u(x,t)$ matches given initial conditions.

Exercise 13.4. How should the coefficients A_n and B_n in (13.4) be picked so that $u(x,t)$ is a solution of the following initial boundary value problem?

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= \sin(\pi x) - \tfrac{1}{9}\sin(3\pi x) + \tfrac{1}{25}\sin(5\pi x) - \cdots, \\ u_t(x,0) &= 0. \end{aligned}$$

Exercise 13.5. The initial position $u(x,0) = f(x)$ given in the previous problem can be written as the infinite series

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin((2k-1)\pi x).$$

Use a computer algebra system or graphing calculator to graph the truncated series

$$f_N(x) = \sum_{k=1}^{N} \frac{(-1)^{k+1}}{(2k-1)^2} \sin((2k-1)\pi x)$$

over the interval $[0,1]$ for $N = 5$, 10, and 20. By looking at the graph of f_N as N increases, what function does $f_N(x)$ appear to be approaching as $N \to \infty$?

At this point it appears that we are limited to only special types of initial conditions. The initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ of the example and problems in this chapter are all of a special form, namely, written explicitly as a combination of the functions $\sin(n\pi x/L)$. This allows the constants A_n and B_n in (13.2) or (13.4) to be selected by inspection in order to match the initial conditions. For a function such as $f(x) = x$ which is not explicitly given in terms of sine functions, we need a method for expanding $f(x)$ as

$$f(x) = a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \cdots = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L).$$

The resulting series is called the *Fourier sine series* of $f(x)$ and will be discussed further in the next chapter.

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Chapter 14

Fourier Series and the Wave Equation

In this chapter it will be shown how to rewrite the initial position $u(x,0) = f(x)$ and velocity $u_t(x,0) = g(x)$ of a string with fixed ends as sums of the string's mode shapes $v_n(x) = \sin(n\pi x/L)$. Then the solution of an initial boundary value problem for the string will be constructed by the principle of superposition.

14.1. Fourier sine series

The task of taking a function f and finding numbers a_n so that

$$\begin{aligned} f(x) &= a_1 \sin(\pi x/L) + a_2 \sin(2\pi x/L) + \cdots \\ &= \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) \end{aligned} \tag{14.1}$$

is one problem within *Fourier analysis.* Fourier analysis is an area of mathematics concerned with the representation of arbitrary functions in terms of certain fundamental functions. Joseph Fourier's contributions in the early 19th century to trigonometric expansions and their applications in physics led to the naming of this branch of mathematics. Additional material on Fourier analysis and its application to partial differential equations can be found in a number of textbooks, including [**Far, Fol**].

The functions f which we will express in the form (14.1) are those which satisfy $f(0) = 0$ and $f(L) = 0$. Thinking in terms of a string with fixed ends and initial position $u(x,0) = f(x)$, the conditions $f(0) = 0$ and $f(L) = 0$ are simply requiring that the ends of the string are fixed at time $t = 0$ as well. A similar restriction $g(0) = g(L) = 0$ for the initial condition $u_t(x,0) = g(x)$ requires that the two ends of the string are not moving at time $t = 0$.

A careful proof that a function $f : [0, L] \to \mathbf{R}$ can be written in the form (14.1) for all $0 \le x \le L$ not only requires $f(0) = f(L) = 0$, but also assumes f is continuous and f' is at least piecewise continuous with jump discontinuities. If f is such a function, then an important result in Fourier analysis shows that there is a unique sequence of numbers $a_1, a_2, a_3, \ldots$ such that $f(x)$ can be written as (14.1) for all x in $[0, L]$. This infinite combination of sine functions is called the **Fourier sine series** expansion of f on the interval $[0, L]$.

The coefficients a_n in the Fourier sine series (14.1) can be computed by exploiting the following property of the trigonometric functions $v_n(x) = \sin(n\pi x/L)$: for any two mode shapes v_n and v_m ($n \neq m$), the integral $\int_0^L v_n(s)v_m(s)ds$ is zero. To verify this, first note that the integral

$$\int_0^L v_n(s)v_m(s)ds = \int_0^L \sin(n\pi s/L)\sin(m\pi s/L)ds$$

can be rewritten using a trigonometric identity as

$$\int_0^L v_n(s)v_m(s)ds = \int_0^L \tfrac{1}{2}\Big[\cos\big((n-m)\pi s/L\big) - \cos\big((n+m)\pi s/L\big)\Big]ds.$$

Evaluating and simplifying the right side when $n \neq m$ shows that

$$\int_0^L v_n(s)v_m(s)ds = 0. \tag{14.2}$$

This property, called *orthogonality*, provides a method for computing the coefficients a_n in the Fourier series (14.1). Writing (14.1) as

$$f(x) = \sum_{n=1}^{\infty} a_n v_n(x)$$

and multiplying both sides by $v_m(x)$ gives

$$f(x)v_m(x) = v_m(x)\sum_{n=1}^{\infty} a_n v_n(x) = \sum_{n=1}^{\infty} a_n v_n(x) v_m(x).$$

Integrating from 0 to L then results in

$$\begin{aligned} \int_0^L f(s)v_m(s)ds &= \int_0^L \sum_{n=1}^{\infty} a_n v_n(s) v_m(s) ds \\ &= \sum_{n=1}^{\infty} a_n \int_0^L v_n(s) v_m(s) ds. \end{aligned}$$

From the orthogonality property (14.2) of the mode shapes, the integral $\int_0^L v_n(s)v_m(s)ds$ is zero for each n which is different from m. The only term in the series which may not be zero is the one for which $n = m$, and so the infinite series reduces to a single term,

$$\int_0^L f(s)v_m(s)ds = a_m \int_0^L v_m(s)v_m(s)ds.$$

Computing the integral on the right side of this equation shows that

$$\int_0^L v_m(s)v_m(s)ds = \int_0^L \sin^2(m\pi s/L)ds = \frac{L}{2},$$

and so the value of the coefficient a_m is given by the expression

$$a_m = \frac{2}{L}\int_0^L f(s)\sin(m\pi s/L)ds. \tag{14.3}$$

Example 14.1. In this example we will write the function

$$f(x) = 1 - |2x - 1|$$

in terms of its Fourier sine series on the interval $[0, 1]$. Note that f is continuous, f' is piecewise continuous (with a jump discontinuity at $x = 1/2$), $f(0) = 0$, and $f(1) = 0$. From Fourier analysis it is then possible to express $f(x)$ as an infinite series $\sum_{n=1}^{\infty} a_n \sin(n\pi x)$ for $0 \le x \le 1$. With $L = 1$ in (14.3), the coefficients a_n are computed by

$$a_n = 2\int_0^1 (1 - |2s - 1|)\sin(n\pi s)ds.$$

This integral can be split into two integrals depending upon whether $2s > 1$ or $2s < 1$ to accommodate the absolute value by

$$\begin{aligned} a_n &= 2\int_0^{1/2} (1 - |2s - 1|) \sin(n\pi s) ds \\ &\quad + 2\int_{1/2}^{1} (1 - |2s - 1|) \sin(n\pi s) ds \\ &= 2\int_0^{1/2} 2s \sin(n\pi s) ds + 2\int_{1/2}^{1} (2 - 2s) \sin(n\pi s) ds. \end{aligned}$$

Integrating by parts in each integral and simplifying results in

$$a_n = \frac{8\sin(n\pi/2)}{n^2\pi^2}.$$

The Fourier sine series expansion of $f(x)$ is then

$$(14.4) \qquad 1 - |2x - 1| = \sum_{n=1}^{\infty} \frac{8\sin(n\pi/2)}{n^2\pi^2} \sin(n\pi x), \quad 0 \le x \le 1.$$

Noting that $\sin(n\pi/2)$ is zero when n is even, one could rewrite this series using only the odd terms by setting $n = 2k - 1$.

The N^{th} **partial sum** of the Fourier sine series for $f(x)$ is defined to be the truncated sum

$$f_N(x) = \sum_{n=1}^{N} a_n \sin(n\pi x/L).$$

This is a sum of a finite number of terms of the Fourier series which provides an approximation of $f(x)$ on the interval $0 \le x \le L$. For the function $f(x)$ in Example 14.1, the partial sums $f_1(x)$ and $f_3(x)$ found by taking one and three terms of the series (14.4) respectively are

$$\begin{aligned} f_1(x) &= \frac{8}{\pi^2} \sin(\pi x), \\ f_3(x) &= \frac{8}{\pi^2} \sin(\pi x) - \frac{8}{9\pi^2} \sin(3\pi x). \end{aligned}$$

Figure 14.1 compares the graph of the function $f(x)$ with the graphs of the partial sums $f_1(x)$, $f_3(x)$ and $f_{10}(x)$.

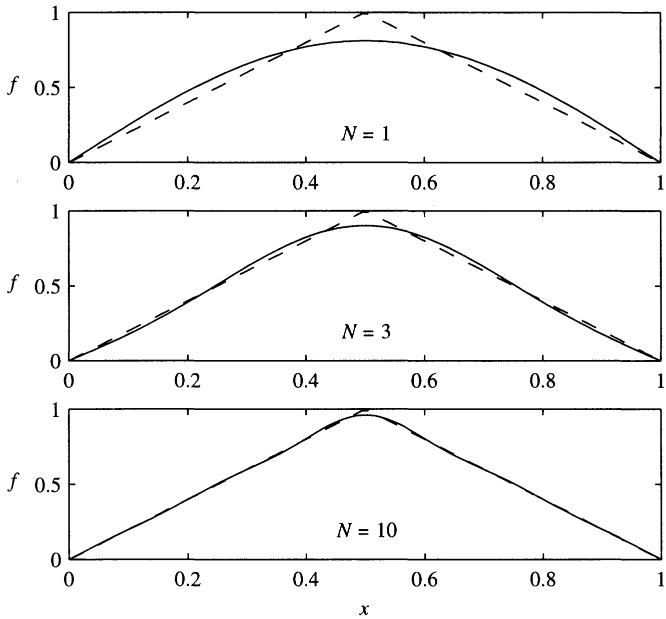

Figure 14.1. Comparing the graph of the function $f(x)$ from Example 14.1 (dashed) with the graph of the N^{th} partial sum of its Fourier sine series (solid).

Similar Fourier series expansions can be made using cosine functions, or a mixture of cosine and sine functions. For example, the mode shapes for a string of length L with both ends free are

$$v_0(x) = 1, \quad v_1(x) = \cos(\pi x/L), \quad v_2(x) = \cos(2\pi x/L), \ldots .$$

An expansion of a function $f(x)$ in terms of these mode shapes is called the **Fourier cosine series** expansion of $f(x)$.

Exercise 14.2. Find the Fourier sine series expansion on the interval $[0, 1]$ for the functions (a) $f(x) = x(1-x)$ and (b) $f(x) = 1-\cos(2\pi x)$. Then use computer generated plots or a graphing calculator to compare the graph of $f(x)$ with its partial Fourier sum $f_3(x)$.

Exercise 14.3. It is possible to extend the idea of Fourier sine series to functions which do not satisfy the boundary conditions $f(0) = 0$ and $f(L) = 0$. If $f : [0, L] \to \mathbf{R}$ is any continuous function for which

f' exists and is continuous, then the Fourier sine series (14.1) with the coefficients a_n computed by (14.3) still converges to $f(x)$ at every point x in the open interval $0 < x < L$. At $x = 0$ and $x = L$, however, the Fourier sine series always sums to 0 since each mode shape $\sin(n\pi x/L)$ is zero at these end points. If $f(0)$ or $f(L)$ is not zero, then (14.1) no longer holds at $x = 0$ or $x = L$.

(a) Find the Fourier sine series expansion for $f(x) = x$ on the interval $[0, 1]$.

(b) By graphing the partial Fourier sums $f_1(x)$, $f_5(x)$, $f_{10}(x)$, and $f_{20}(x)$, for which value(s) of x in the interval $[0, 1]$ does it appear that the Fourier series will not converge to $f(x) = x$?

Exercise 14.4. The companion MATLAB software (see page xiii) includes two scripts for plotting Fourier series. The script `wvfour1` is a graphical interface for quick plotting of partial sums of Fourier sine or cosine series of a function. The script `wvfour2` is a graphical method for choosing values of the coefficients a_n in the Fourier sine series expansion for $f(x)$. In MATLAB, run each script and construct Fourier series of the functions $f(x) = \sin(2\pi x)$, $f(x) = x(1 - x)$, and $f(x) = x$.

14.2. Fourier series solution of the wave equation

The ability to compute the Fourier sine series of a function now gives us a method for rewriting the initial position $u(x, 0) = f(x)$ and initial velocity $u_t(x, 0) = g(x)$ of a string in terms of mode shapes of the string. Consider the following initial boundary value problem for the string:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad 0 < x < L, 0 < t < \infty, && \text{(PDE)}\\
u(0, t) &= 0, && \text{(BC)}\\
u(L, t) &= 0, && \\
u(x, 0) &= f(x), && \text{(IC)}\\
u_t(x, 0) &= g(x). &&
\end{aligned}
\tag{14.5}
$$

To find a solution of this problem, we can start by superimposing the string's modes of vibration to write down a general solution of the wave equation which satisfies the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ as

$$(14.6)\quad u(x,t) = \sum_{n=1}^{\infty}[A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)] \sin(n\pi x/L).$$

The next step is to pick the arbitrary constants A_n and B_n so that $u(x,0)$ and $u_t(x,0)$ match the initial conditions (IC) given in (14.5).

The constants A_n and B_n are found by first rewriting the initial position and velocity in terms of the mode shapes $\sin(n\pi x/L)$. Suppose that the initial position and velocity in (14.5) have been expressed in terms of their Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L), \quad g(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L).$$

Matching the initial position $u(x,0)$ from (14.6) with the Fourier series of $f(x)$ gives the condition

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L).$$

This shows that $A_n = a_n$, the n^{th} Fourier sine coefficient of $f(x)$. To find the values of B_n, note that the derivative $u_t(x,t)$ of the superposition (14.6) is

$$u_t(x,t) = \sum_{n=1}^{\infty}\Big[-\frac{n\pi c}{L}A_n \sin(n\pi ct/L) + \frac{n\pi c}{L}B_n \cos(n\pi ct/L)\Big] \sin(n\pi x/L).$$

Matching $u_t(x,0)$ with the Fourier series expansion of the initial velocity $g(x)$ results in

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin(n\pi x/L) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L).$$

This shows that $(n\pi c/L)B_n = b_n$, and so $B_n = (b_n L)/(n\pi c)$. Now that we have found the constants A_n and B_n, the solution of the initial boundary value problem (14.5) can be written as

(14.7)

$$u(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos(n\pi ct/L)+\frac{b_nL}{n\pi c}\sin(n\pi ct/L)\right]\sin(n\pi x/L).$$

Example 14.5. Consider the displacement $u(x,t)$ of a string with fixed ends,

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0<x<1,\ t>0,\\ u(0,t) &= 0,\\ u(1,t) &= 0, \end{aligned}$$

whose initial position and velocity are

$$\begin{aligned} u(x,0) &= 0,\\ u_t(x,0) &= 1-|2x-1|\,. \end{aligned}$$

As a superposition of mode shapes, the solution $u(x,t)$ will have the form (14.7) with string length $L=1$ and wave speed $c=1$, so

$$u(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos(n\pi t)+\frac{b_n}{n\pi}\sin(n\pi t)\right]\sin(n\pi x). \tag{14.8}$$

The initial position is already in terms of the mode shapes $\sin(n\pi x)$ of the string, since

$$u(x,0)=0=\sum_{n=1}^{\infty}a_n\ \sin(n\pi x),$$

with $a_n=0$ for all n. The initial velocity was written in terms of $\sin(n\pi x)$ in Example 14.1 as

$$u_t(x,0)=1-|2x-1|=\sum_{n=1}^{\infty}b_n\sin(n\pi x),$$

with $b_n=8\sin(n\pi/2)/(n\pi)^2$. Substituting $b_n=8\sin(n\pi/2)/(n\pi)^2$ and $a_n=0$ into (14.8) gives

$$u(x,t)=\sum_{n=1}^{\infty}\frac{8\sin(n\pi/2)}{n^3\pi^3}\sin(n\pi t)\sin(n\pi x) \tag{14.9}$$

as the profile of the string at time t.

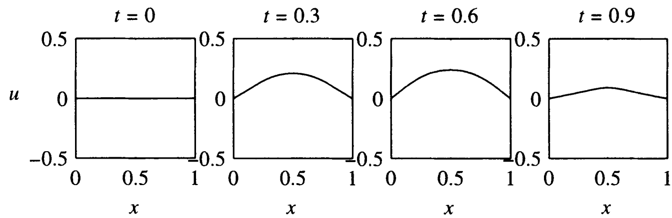

Figure 14.2. Animation of the string represented by Example 14.5.

Note that the initial boundary value problem in the last example represents a fixed string which initially lies in its equilibrium position with $u(x, 0) = 0$. At time $t = 0$, the string is given an initial "kick" upward with velocity $u_t(x, 0) = 1 - |2x - 1|$ at position x. An animation of the resulting displacement $u(x, t)$ can be made by taking a partial sum of the series representation of $u(x, t)$. Using the first twenty terms in the sum (14.9), four frames of animation of the partial sum

$$u_{20}(x, t) = \sum_{k=1}^{20} \frac{8 \sin(n\pi/2)}{n^3\pi^3} \sin(n\pi t) \sin(n\pi x)$$

are shown in Figure 14.2.

Exercise 14.6. Consider the displacement of a vibrating string with fixed ends given by

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u(0, t) &= 0, \\ u(1, t) &= 0. \end{aligned}$$

Use the Fourier sine series expansions found in Exercise 14.2 to write down the series solution of the vibrating string with the following initial conditions:

(a) $u(x, 0) = x(1 - x), \quad u_t(x, 0) = 0.$

(b) $u(x, 0) = 0, \quad u_t(x, 0) = x(1 - x).$

Animate the solutions by approximating $u(x, t)$ with a partial sum of the series solution for $u(x, t)$.

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Part 3

Waves in Conservation Laws

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Chapter 15

Conservation Laws

While the wave equation has many solutions which illustrate waves and their properties, wave behavior can be found in applications which are modeled by other partial differential equations. In the following chapters we will look at a class of mathematical models which are derived from *conservation laws.* Later it will be shown that many of these models possess solutions with wave behavior.

15.1. Derivation of a general scalar conservation law

A **conservation law** is an equation which accounts for all of the ways that the amount of a particular quantity can change. This accounting is one of the basic principles of mathematical modeling and can be applied to a variety of quantities such as mass, momentum, energy, and population.

Suppose that a medium, essentially one-dimensional and positioned along the $x-$axis, contains some substance which can move or flow. This quantity could be, for example, cars moving along a section of road, particles of pollutant in a narrow stream of water, or heat energy flowing along a wire. For brevity, let Q represent this quantity (cars, particles, energy, etc...). In this section we will derive

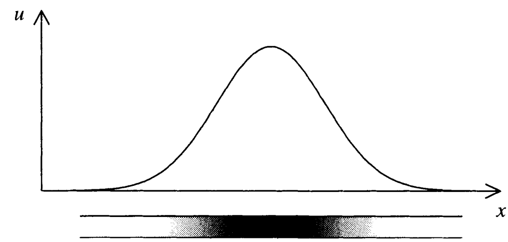

Figure 15.1. Profile at time t of the density $u(x,t)$ of a quantity in a one-dimensional medium.

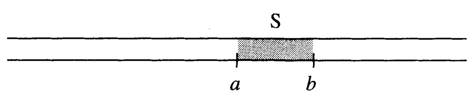

Figure 15.2. A short segment S of the medium over the interval $[a,b]$.

a general conservation law which describes the amount of Q in the medium at time t.

Let $u(x,t)$ measure the density or concentration (amount per unit length) of Q at position x of the medium at time t (Figure 15.1). The value of u could indicate, for example, the density of traffic (cars per mile) or concentration of pollutant (grams per meter) at position x.

Now let S be any small segment of the medium with endpoints located at $x = a$ and $x = b$ with $a < b$ (Figure 15.2). It will be assumed that changes in the amount of the quantity Q within this segment can occur in only two ways: either Q enters or leaves S through its ends at $x = a$ and $x = b$, or Q is somehow being added (created) or removed (destroyed) from the medium within the segment. By accounting for all of the ways in which the total amount of Q can change within S, we are forming a general conservation law for Q:

(15.1)
$$\boxed{\begin{array}{l}\text{The net (time) rate}\\ \text{of change of the total}\\ \text{amount of } Q \text{ in } S\end{array}} = \boxed{\begin{array}{l}\text{The rate at which } Q \text{ enters}\\ \text{or leaves } S \text{ through the left}\\ \text{end } x = a\end{array}} + \boxed{\begin{array}{l}\text{The rate at which } Q \text{ en-}\\ \text{ters or leaves } S \text{ through}\\ \text{the right end } x = b\end{array}} + \boxed{\begin{array}{l}\text{The rate at which } Q \text{ is cre-}\\ \text{ated or removed within } S\end{array}}.$$

The next step will be to quantify the different parts of this conservation principle.

Since $u(x,t)$ is the amount of Q per unit length along the medium, the total amount of Q in the segment S at time t is computed by the integral $\int_a^b u(x,t)dx$. As the quantity Q flows through the medium, the amount of Q within S can change over time; the rate at which this amount changes with respect to time is given by the derivative

$$\frac{d}{dt}\int_a^b u(x,t)dx. \tag{15.2}$$

The rate at which Q enters S through either of its ends will be described by a **flux function**. Let $\phi(x,t)$ denote the rate (amount per unit time) at which Q is flowing past position x at time t. A positive value $\phi(x,t) > 0$ indicates that the flow is in the direction of increasing x, while $\phi(x,t) < 0$ means the flow is in the opposite direction. Such a function is called the **flux**. The rate at which Q enters S through the end $x = a$ is then $\phi(a,t)$. If $\phi(a,t)$ is positive, then Q is flowing into S through the left end at $x = a$, while $\phi(a,t) < 0$ indicates Q is flowing out of S through the left end. Similarly, the rate at which Q enters S through the right end at $x = b$ is $-\phi(b,t)$. The extra minus sign at $x = b$ is needed since $\phi(b,t) > 0$ indicates Q is flowing to the right at $x = b$, which decreases (negative rate) the amount of Q in the segment S (see Figure 15.2). The net rate at which Q enters S through its ends is then given by

$$\phi(a,t) - \phi(b,t). \tag{15.3}$$

The addition or removal of Q within the segment S will be represented by a *source function.* Let $f(x,t)$ be the rate (amount per unit time per unit length) at which Q is being added to or removed from the medium at position x and time t. Such a function f is called a **source function**. A positive value $f(x,t) > 0$ indicates that Q is being created or added to the medium at position x, while $f(x,t) < 0$ means Q is being destroyed or removed. The total rate (amount per unit time) at which Q is being created within the segment S at time t is

$$\int_a^b f(x,t)dx. \tag{15.4}$$

Substituting the measurements (15.2), (15.3), and (15.4) into the conservation principle (15.1) results in an equation called a **conservation law in integral form**:

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)dx.} \tag{15.5}$$

An alternative form of the integral conservation law can be derived when u and ϕ are assumed to have continuous first derivatives. With this assumption (15.5) can be rewritten as

$$\int_a^b u_t(x,t)dx = -\int_a^b \phi_x(x,t)dx + \int_a^b f(x,t)dx,$$

so that

$$\int_a^b \big(u_t(x,t) + \phi_x(x,t) - f(x,t)\big)dx = 0.$$

If u_t, ϕ_x, and f are all continuous, then the fact that this integral is zero for *every* $a < b$ along the medium implies that the integrand $u_t + \phi_x - f$ must be zero. This results in a **conservation law in differential equation form**:

$$\boxed{u_t + \phi_x = f.} \tag{15.6}$$

15.2. Constitutive equations

The conservation law (15.6) is a very general equation which relates three functions: the density function u, the flux ϕ, and the source term f. It simply states that the rate of change of the amount of Q at position x depends on the rate at which Q flows past x (flux) and the rate at which Q is created at x (source). In order to determine $u(x,t)$, more must be known about the flux ϕ and the source term f.

The source term f is usually determined or specified from the particular physical problem behind the conservation law. In many cases, it is zero.

Even when $f = 0$, $u_t + \phi_x = 0$ is still only one differential equation for two unknowns u and ϕ. A second equation relating u and ϕ is often given, based on an assumption about the physical process being modeled or on experimental evidence. Such an equation is called a **constitutive equation**. In general, our models will consist of two parts,

$u_t + \phi_x = f$	Conservation Law (Fundamental law of nature),
Relationship between u and ϕ	Constitutive Equation (Approximation based on experience).

The flux ϕ often depends on u. For example, if the rate (amount per time) at which the quantity Q flows past a point depends on the concentration of Q, then the flux is a function of density and forms an explicit constitutive law $\phi = \phi(u)$. When this is the case, the chain rule gives $\phi_x = \phi'(u)u_x$, so that the conservation law (15.6) can be written as

$$\boxed{u_t + \phi'(u)u_x = f.} \tag{15.7}$$

Example 15.1. The inviscid Burgers equation

$$u_t + uu_x = 0$$

is an example of a conservation law in the form (15.7). In this equation the source term $f(x,t)$ is zero and the flux ϕ is a function of u for which $\phi'(u) = u$. One possibility for the flux term is the constitutive equation

$$\phi(u) = \tfrac{1}{2}u^2.$$

The inviscid Burgers equation can then be written in conservation law form,

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0.$$

Exercise 15.2. Write the following equations in conservation law form $u_t + \phi_x = 0$ and identify the flux ϕ:

(a)	$u_t + cu_x = 0$	Advection or convection equation
(b)	$u_t + uu_x - Du_{xx} = 0$	Burgers' equation
(c)	$u_t + uu_x + u_{xxx} = 0$	KdV equation

Exercise 15.3. Sometimes there are alternative ways of writing conservation laws depending upon which terms are considered fluxes and which terms are considered sources. What are the suggested flux and source terms of the Burgers equation when it is written in the forms $u_t + uu_x = Du_{xx}$ and $u_t + uu_x - Du_{xx} = 0$?

Chapter 16

Examples of Conservation Laws

In this chapter, conservation laws and constitutive equations are formulated for the physical applications of a plug flow reactor, diffusion, and traffic flow.

16.1. Plug flow chemical reactor

A plug flow chemical reactor consists of a long tube in which a chemical product A is fed into one end, a chemical reaction takes place within the tube, and a resulting chemical product B is forced out of the opposite end (Figure 16.1). We will follow [**Log**, p. 104] to formulate a conservation law for the amount and distribution of chemical A within the reactor tube at time t.

Let $a(x,t)$ denote the concentration (mass per unit length) of chemical A at position x along the tube and time t. From the previous chapter (Section 15.1), a general conservation law for $a(x,t)$ will have the form

$$a_t + \phi_x = f \tag{16.1}$$

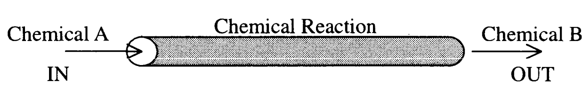

Figure 16.1. A plug flow chemical reactor.

with flux ϕ and source f. In this application, the flux $\phi(x,t)$ gives the rate (mass per unit time) at which chemical A passes by position x at time t. The source $f(x,t)$ describes the rate (mass/time per unit length) at which chemical A is removed from the medium at position x by the chemical reaction.

To determine ϕ, we will make a basic assumption that the chemical mixture within the tube is moving from left to right with a constant speed of c (length per unit time). In this setting, the movement of the mixture carries chemical A past position x at a rate of

$$\phi(x,t) = ca(x,t)$$

with units mass per unit time. The equation $\phi = ca$ is an example of a constitutive equation, an equation relating the flux ϕ and density a based on modeling assumptions. This type of flux function describes an *advection* or *convection* process in which changes in the value of $a(x,t)$ are due to movement of the medium.

Within the tube, the chemical reaction removes chemical A in the process of forming chemical B. Here it will be assumed that the rate at which this occurs is simply a percentage of the concentration of A present, or

$$f(x,t) = -ka(x,t)$$

mass/time per unit length of the tube where $k > 0$ is a constant. The extra minus sign reflects the fact that the amount of chemical A is decreasing.

With the flux $\phi(x,t) = ca(x,t)$ and source $f(x,t) = -ka(x,t)$, the basic conservation law (16.1) becomes

$$\boxed{a_t(x,t) + ca_x(x,t) = -ka(x,t).}$$

Exercise 16.1. Give a physical interpretation of the following initial boundary value problem for the plug flow reactor:

$$\begin{aligned} a_t(x,t) + ca_x(x,t) &= -ka(x,t), \quad 0 < x < L,\ t > 0, \\ a(x,0) &= 0, \\ a(0,t) &= g(t). \end{aligned}$$

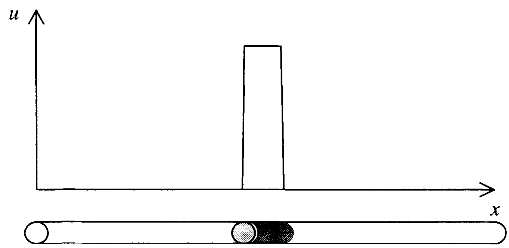

Figure 16.2. Profile at time t of the density $u(x, t)$ of a pollutant in a water pipe with a highly polluted section.

16.2. Diffusion

A second example of a conservation law will be formed to track a pollutant spreading through water by diffusion. Suppose a long pipe full of stagnant water becomes contaminated in a small section due to a temporary break in the pipe (Figure 16.2). Letting $u(x, t)$ be the concentration (mass per unit length) of the pollutant at position x and time t, a general conservation law for $u(x, t)$ will take the form

$$u_t + \phi_x = f.$$

As in the previous example, assumptions based on the physical problem will be needed to formulate ϕ and f.

In this application, the flux $\phi(x, t)$ represents the rate (mass per unit time) at which pollutant is passing by position x at time t. Since the water within the pipe is not moving, the primary method that the pollutant can flow through the water is by *diffusion*. In general, a pollutant will flow from regions of high concentration to regions of lower concentration in an effort to redistribute itself uniformly. For example, suppose the concentration $u(x, t)$ along the pipe at time t has the profile shown in Figure 16.3. If x is a position along the pipe for which the slope $u_x(x, t)$ is positive, then the pollutant should flow left towards a region of lower concentration. Similarly, if x is a position for which $u_x(x, t)$ is negative, then the pollutant should flow to the right (Figure 16.3). A fundamental formulation of this phenomena is Fick's First Law of Diffusion,

$$\phi = -Du_x.$$

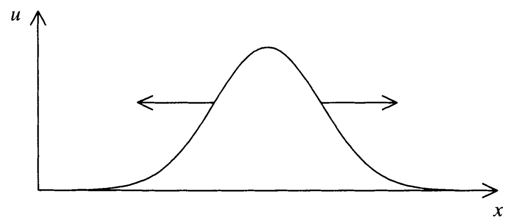

Figure 16.3. Pollutant flows from regions of higher contamination to lower contamination.

Here D is a positive constant, and the extra minus sign indicates that the flow direction is opposite in sign from the slope u_x of the graph of u. This will be our constitutive equation.

The source function $f(x,t)$ represents the rate (mass/time per unit length) at which the pollutant is either entering or being removed from the pipe at position x. If we assume that the break in the pipe has been fixed so that no additional pollutant enters the pipe, and no filtering or chemical reaction takes place which removes the pollutant, then $f = 0$.

With $\phi = -Du_x$ and $f = 0$, the conservation law $u_t + \phi_x = f$ takes the form of the diffusion equation,

$$\boxed{u_t - Du_{xx} = 0.}$$

Exercise 16.2. Give a physical interpretation of the following initial boundary value problem in terms of pollutant contaminating a pipe full of stagnant water:

$$\begin{aligned} u_t - Du_{xx} &= 2, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \\ \lim_{x \to \pm\infty} u(x,t) &= 0. \end{aligned}$$

Exercise 16.3. Suppose that the water in the pipe is not stagnant, but rather moves in one direction with a constant speed c. How does the form of the flux ϕ change? Note that the flux now consists of two parts, diffusion and convection. What is the resulting conservation law?

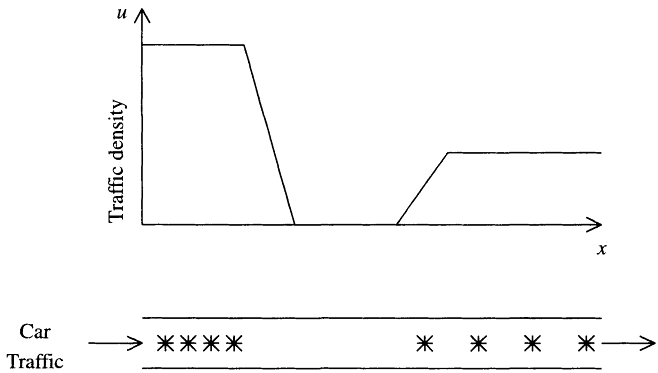

Figure 16.4. Continuous representation of traffic density along a single lane road.

16.3. Traffic flow

Since its first developments in the mid 1950's by M.J. Lighthill and G.B. Whitham [**LW**] and P.I. Richards [**R**], the deterministic modeling of traffic flow has yielded several examples of wave behavior. Here we will follow parts of the books by Haberman [**Hab1**] and Whitham [**Whi**] to form conservation laws which model traffic flow, and later observe wave phenomena arising from these models.

As a simplified example, consider automobile traffic moving along a section of single lane road with no exits or entrances. Let $u(x,t)$ represent the density of cars (number of cars per mile) at position x along the road at time t. The function $u(x,t)$ in principle should be a discrete valued function since cars are discrete objects; however, we will assume that $u(x,t)$ is a continuous representation of the traffic density such as the one shown in Figure 16.4. As before, the basic conservation law for the traffic density $u(x,t)$ is

$$u_t + \phi_x = f.$$

In this conservation law, the source $f(x,t)$ represents the rate (cars/hour per mile) at which cars are added or removed from the road at position x. With the assumption that there are no exits or

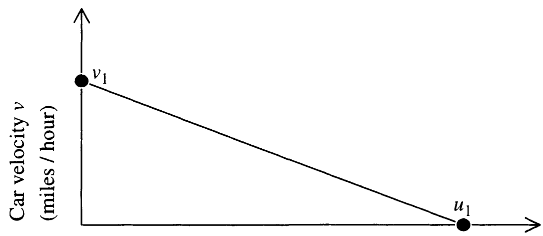

Figure 16.5. Higher traffic density generally results in lower traffic speed.

entrances to the road and that cars do not appear or disappear from the road for any other reason, the source function $f(x,t)$ is zero.

The flux function $\phi(x,t)$ represents the rate (cars per hour) at which cars are passing position x along the road at time t. To an observer standing along the side of a road, the rate at which cars pass by depends not only on the traffic density u, but also on the traffic velocity v. If v is measured in miles per hour, then the flux ϕ is the product

$$\phi = u \text{ (cars/mile)} \times v \text{ (miles/hour)} = uv \text{ (cars/hour)}.$$

Traffic velocity v is generally not constant and is related to factors such as traffic density, weather, and time of day. As a simple model, we will assume that the velocity v of the cars depends only on the traffic density, and in particular, denser traffic results in lower speeds. Suppose that drivers will travel at a maximum speed of v_1 miles per hour on a road which has little or no traffic ($u = 0$). We will also assume that traffic is at its maximum density u_1 cars per mile when the cars have come to a complete stop ($v = 0$). A linear model of this connection between traffic velocity and traffic density is shown in Figure 16.5 and is described by the equation

$$v = v_1 - \frac{v_1}{u_1}u, \quad 0 \le u \le u_1.$$

The constitutive equation relating flux ϕ and traffic density u is then

$$\phi = uv = v_1(u - u^2/u_1) \quad \text{(cars/hour)}. \tag{16.2}$$

With the car flux (16.2) and source function $f(x,t) = 0$, the conservation law $u_t + \phi_x = f$ modeling traffic density along the road becomes

$$\boxed{u_t + v_1(1 - 2u/u_1)u_x = 0.}$$

Exercise 16.4. By estimating an average car length, determine a value for u_1, the maximum possible density of cars (cars per mile) along a stretch of single lane road.

Exercise 16.5. Another model of traffic velocity as a function of traffic density is $v = k\ln(u_1/u)$ where k is a positive constant. By making a qualitative graph of $v = k\ln(u_1/u)$, tell how driver behavior described by this model differs from the driver behavior reflected in the linear model shown in Figure 16.5. Using this model of velocity, what is the resulting flux ϕ and conservation law for traffic density $u(x,t)$?

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Chapter 17

The Method of Characteristics

The concept of *characteristics* will be reintroduced in this chapter as a method for constructing a solution of an initial value problem of the form

$$
\begin{aligned}
&u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0, \\
&u(x,0) = u_0(x).
\end{aligned}
\tag{17.1}
$$

Similar to characteristic lines for the wave equation, characteristics are special curves in the xt−plane which transmit the given initial profile $u(x,0)$ forward in time. Characteristics will be used to solve initial value problems of the form (17.1) for the advection equation, general linear conservation laws, and nonlinear conservation laws.

17.1. Advection equation

If the constitutive equation relating a density u and flux ϕ is of the form $\phi(x,t) = cu(x,t)$ for some constant c, then the initial value problem (17.1) becomes

$$u_t + cu_x = 0, \quad -\infty < x < \infty,\ t > 0, \tag{17.2a}$$

$$u(x,0) = u_0(x). \tag{17.2b}$$

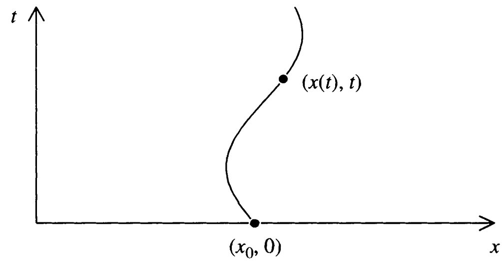


Figure 17.1. A curve $(x(t), t)$ in the $xt-$plane.

The conservation law here is the *advection equation*, and the initial condition gives us the profile of the solution $u(x,t)$ at time $t = 0$. We will now describe the method of characteristics for finding the value of the solution u at points (x,t) for a later time $t > 0$.

The method of characteristics uses special curves in the $xt-$plane along which the partial differential equation (17.2a) becomes an ordinary differential equation. Suppose $(x(t), t)$ is a curve in the $xt-$plane starting from the point $(x_0, 0)$ on the $x-$axis (Figure 17.1). As $(x(t), t)$ moves along this curve, the value of $u(x(t), t)$ changes at the rate of $\frac{d}{dt}u(x(t), t)$. By the chain rule, this derivative is

$$\frac{d}{dt}u(x(t),t) = u_x(x(t),t)\frac{dx}{dt} + u_t(x(t),t). \tag{17.3}$$

The right hand side $u_t + \frac{dx}{dt}u_x$ resembles $u_t + cu_x$, part of the conservation law (17.2a). In fact, if we select the curve $(x(t), t)$ so that

$$\frac{dx}{dt} = c,$$

the chain rule (17.3) and the conservation law (17.2a) give

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + cu_x(x(t),t) = 0.$$

This implies that the value of u is constant along this particular curve, and so the value of u at each point on the curve is the same as the value of u at the initial point $(x_0, 0)$. From the initial condition (17.2b), this value is seen to be $u_0(x_0)$.

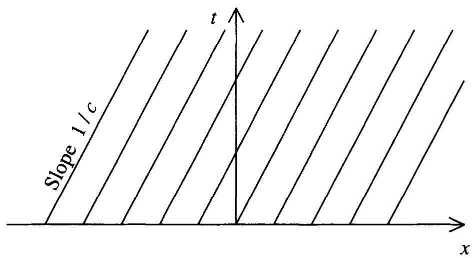


Figure 17.2. Characteristics $x - ct = x_0$ of the advection equation $u_t + cu_x = 0$.

The special curve $(x(t), t)$ starting at the point $(x_0, 0)$ is determined by the conditions

$$\frac{dx}{dt} = c, \quad x(0) = x_0.$$

Solving this initial value problem shows that $x(t)$ is given by

$$x = ct + x_0.$$

This curve is called a **characteristic curve** or **characteristic** of the equation $u_t + cu_x = 0$. As shown in Figure 17.2, the characteristic curves here are parallel lines in the xt–plane, each with slope $1/c$ but starting at different initial points $(x_0, 0)$ on the x–axis. The derivative dx/dt is called the **speed** of the characteristics, which for this problem is c.

Using the fact that u is constant along the lines $x = ct + x_0$, we can construct the value of $u(x, t)$ at any point (x, t). Given a point (x, t), a characteristic line extends back from (x, t) to the point $(x_0, 0)$ on the x–axis where x_0 is given by $x_0 = x - ct$. Since the function $u(x, t)$ is known to be constant along a characteristic, the value of u at (x, t) is the same as the value of u at $(x_0, 0)$. By the initial condition (17.2b), this value is

$$u(x, t) = u(x_0, 0) = u_0(x_0) = u_0(x - ct).$$

The solution $u(x, t) = u_0(x - ct)$ of (17.2) is a traveling wave with initial profile $u_0(x)$ transmitted through the medium with velocity c.

Example 17.1. Consider the initial value problem for the advection equation

$$u_t + 4u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = \arctan(x).$$

Along a curve $(x(t),t)$, the derivative of $u(x(t),t)$ is

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + \frac{dx}{dt}u_x(x(t),t).$$

Picking $x(t)$ to satisfy

$$\frac{dx}{dt} = 4, \quad x(0) = x_0$$

results in the characteristic curve $x = 4t + x_0$. Along this curve,

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + 4u_x(x(t),t) = 0,$$

so $u(x(t),t)$ has a constant value along $x = 4t+x_0$. Now at any point (x,t), the characteristic line through (x,t) extends back to the point $(x_0,0)$ on the x−axis where $x_0 = x - 4t$. Since u is constant along this characteristic, the value of u at (x,t) is

$$u(x,t) = u(x_0,0) = \arctan(x_0) = \arctan(x-4t).$$

The solution of this initial value problem is a traveling wave with profile $\arctan(x)$ moving with speed 4.

Exercise 17.2. Use characteristics to find the solution of

$$u_t + 2u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = e^{-x^2}.$$

Exercise 17.3. Consider the following initial boundary value problem for the advection equation,

$$\begin{aligned} u_t + 2u_x &= 0, && x > 0,\ t > 0,\\ u(x,0) &= 0, && x \geq 0,\\ u(0,t) &= \frac{t}{1+t^2}, && t \geq 0. \end{aligned}$$

In this problem the value of u is known at points $(x_0,0)$ and $(0,t_0)$ along the positive parts of the x and t axes (Figure 17.3).

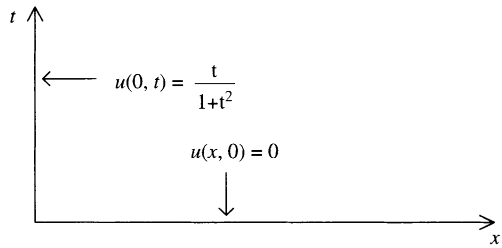


Figure 17.3. Beginning of an $xt-$diagram for Exercise 17.3.

(a) Find the equation for characteristics which start at $(x_0, 0)$ along the positive part of the $x-$axis. At any (x, t) in the first quadrant with $x > 2t$, what is the value of $u(x, t)$?

(b) Find the equation for characteristics which start at $(0, t_0)$ along the positive part of the $t-$axis. At any (x, t) in the first quadrant with $x < 2t$, what is the value of $u(x, t)$?

(c) Sketch the profile of the solution $u(x, t)$ at times $t = 0, 1, 2, 3$.

17.2. Nonhomogeneous advection equation

The use of characteristics for constructing solutions of the nonhomogeneous advection equation

$$u_t + cu_x = f \tag{17.4}$$

is similar to the method for the homogeneous equation $u_t + cu_x = 0$. As before, the rate of change of u along a curve $(x(t), t)$ is

$$\frac{d}{dt}u(x(t), t) = u_t(x(t), t) + \frac{dx}{dt}u_x(x(t), t),$$

and the characteristic curve starting at $(x_0, 0)$, found by solving

$$\frac{dx}{dt} = c, \quad x(0) = x_0,$$

is $x = ct + x_0$. The rate of change of u along a characteristic for the nonhomogeneous equation (17.4) is then given by

$$\frac{d}{dt}u(x(t), t) = u_t(x(t), t) + cu_x(x(t), t) = f(x(t), t).$$

The value of u along the characteristic is not constant, but rather is found by solving the initial value problem

$$\frac{dU}{dt} = f, \quad U(0) = u_0(x_0)$$

where $U(t) = u(x(t), t)$ is the value of u along the curve $x = ct + x_0$.

Example 17.4. Consider the initial value problem

$$\begin{aligned} u_t + 4u_x &= 1, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \arctan(x). \end{aligned}$$

The characteristic curve starting at the point $(x_0, 0)$, found by solving $dx/dt = 4$, is $x = 4t + x_0$. The rate of change of $u(x(t), t)$ along this curve is

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + 4u_x(x(t),t) = 1$$

since $u_t + 4u_x = 1$. Integrating with respect to t gives

$$u(x(t), t) = t + A$$

for some constant of integration A. To find A, note that at $t = 0$,

$$A = u(x(0), 0) = u(x_0, 0) = \arctan(x_0).$$

The value of u along the characteristic curve beginning at $(x_0, 0)$ is then $u(x(t), t) = t + \arctan(x_0)$. Now at a point (x, t), the characteristic through (x, t) extends back to the point $(x_0, 0)$ on the $x-$axis where $x_0 = x - 4t$. The value of u at $(x_0, 0)$ then gives the value of u at (x, t) by

$$u(x,t) = t + \arctan(x_0) = t + \arctan(x - 4t).$$

Exercise 17.5. Use characteristics to find the solution of

$$\begin{aligned} u_t + 2u_x &= -u, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \frac{1}{1+x^2}. \end{aligned}$$

17.3. General linear conservation laws

Characteristics can also be used to solve initial value problems for linear conservation laws in the form

$$u_t + c(x,t)u_x = 0, \quad -\infty < x < \infty,\ t > 0, \tag{17.5a}$$

$$u(x,0) = u_0(x). \tag{17.5b}$$

The coefficient $c(x,t)$ may not be constant; nevertheless the characteristic starting at the point $(x_0, 0)$ in the xt–plane is still found by solving

$$\frac{dx}{dt} = c(x,t), \quad x(0) = x_0. \tag{17.6}$$

Since the characteristic speed $c(x,t)$ is not necessarily constant, the characteristics are not necessarily lines. The value of $u(x(t),t)$, however, is still constant along these curves since

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + c(x,t)u_x(x(t),t) = 0$$

by (17.5a) and (17.6). As with the advection equation, the value of u at a point (x,t) can found by:

(1) Finding the characteristic curves by solving $dx/dt = c(x,t)$, $x(0) = x_0$;

(2) Finding the particular initial point $(x_0, 0)$ for the characteristic passing through (x,t);

(3) Using x_0 to compute $u(x,t) = u(x_0,0) = u_0(x_0)$.

Exercise 17.6. Consider the following initial value problem:

$$u_t + txu_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \frac{1}{1+x^2}.$$

(a) Solve (17.6) with $c(x,t) = xt$ to show that the equation of the characteristic curve starting at $(x_0, 0)$ is $x = x_0 e^{t^2/2}$. Plot several characteristics in the xt–plane for $0 \le t \le 2$ by picking at least 10 different starting points $(x_0, 0)$ with $-5 \le x_0 \le 5$.

(b) Find the solution $u(x,t)$ of the initial value problem and animate the result.

17.4. Nonlinear conservation laws

Returning to the general initial value problem (17.1), suppose now that a constitutive equation relating u and ϕ implies that ϕ is a function of u, $\phi = \phi(u)$. By the chain rule, the general conservation law $u_t + \phi_x = 0$ becomes $u_t + \phi'(u)u_x = 0$, or by letting $c(u) = \phi'(u)$,

$$u_t + c(u)u_x = 0.$$

Unless $c(u)$ is constant, this is a nonlinear equation for u.

In this section we will use the method of characteristics to solve initial value problems of the form

$$\begin{aligned} u_t + c(u)u_x &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

The characteristic starting at $(x_0, 0)$ for this problem is found by solving

$$\frac{dx}{dt} = c(u(x,t)), \quad x(0) = x_0. \tag{17.7}$$

At this stage, however, we do not know the function $u(x,t)$ needed to complete this ordinary differential equation. Whatever the characteristic ends up being, the value of $u(x,t)$ is still constant along the resulting curve $(x(t), t)$ since

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= u_t(x(t),t) + \frac{dx}{dt}u_x(x(t),t) \\ &= u_t(x(t),t) + c(u(x(t),t))u_x(x(t),t) = 0. \end{aligned}$$

As before, this shows that the value of u along a characteristic is constant. The value of u at each point along the curve is then the same as the value of u at the starting point $(x_0, 0)$ of the curve, which by the initial condition is

$$u(x,t) = u(x_0, 0) = u_0(x_0).$$

Now that it is known u is constant along a characteristic, we can return to (17.7) to solve for $x(t)$. Since $u(x,t)$ has the constant value $u_0(x_0)$ along the characteristic starting at $(x_0, 0)$, we can write the initial value problem (17.7) as

$$\frac{dx}{dt} = c(u_0(x_0)), \quad x(0) = x_0$$

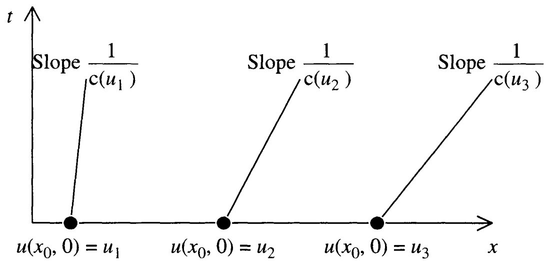


Figure 17.4. Characteristics of a nonlinear conservation law $u_t + c(u)u_x = 0$ may not be parallel.

and solve it to find that

$$x = c(u_0(x_0))t + x_0. \tag{17.8}$$

Like the advection equation, the characteristics are lines. Unlike the advection equation, however, the characteristics are not necessarily parallel since the slope $1/c(u_0(x_0))$ of each line depends on the value of u at the initial point of the curve (see Figure 17.4). The procedure for finding the value of u at a point (x, t) is now:

(1) Construct the characteristics $x = c(u_0(x_0))t + x_0$ using the speed $c(u)$ from $u_t + c(u)u_x = 0$ and initial profile $u_0(x_0)$.

(2) Find the initial point $(x_0, 0)$ of the characteristic passing through (x, t) by solving $x = c(u_0(x_0))t + x_0$ for x_0. This step often has to be done numerically.

(3) Use x_0 to calculate $u(x, t) = u(x_0, 0) = u_0(x_0)$.

Example 17.7. If $\phi = \frac{1}{2}u^2$, then the conservation law $u_t + \phi_x = 0$ takes the form of the inviscid Burgers equation $u_t + uu_x = 0$ with characteristic speed $c(u) = u$. Now consider the initial value problem

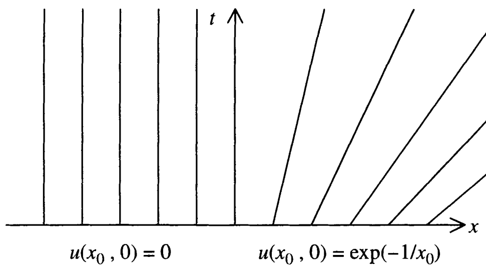
$$u_t + uu_x = 0, \quad -\infty < x < \infty, \; t > 0,$$

$$u(x, 0) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

Since $c(u_0(x_0)) = u_0(x_0)$, the characteristic starting at $(x_0, 0)$ is the line $x = u_0(x_0)t + x_0$. The characteristic lines for this initial value problem are then

$$x = 0 \cdot t + x_0 \qquad \text{if } x_0 \le 0,$$
$$x = \left(e^{-1/x_0}\right) t + x_0 \quad \text{if } x_0 > 0.$$

Note that as the starting point $(x_0, 0)$ moves further to the right along the x–axis, the characteristic slope $1/c = e^{1/x_0}$ decreases to zero:



At a point (x, t), the value of u is found from $u(x, t) = u(x_0, 0)$, where $(x_0, 0)$ is the starting point of the characteristic passing through (x, t). When $x \le 0$, the characteristic passing through (x, t) is vertical and starts at the point $(x, 0)$. In this case, the initial condition gives $u(x, t) = u(x, 0) = 0$ since $x \le 0$.

When $x > 0$, the characteristic line passing through (x, t) has its starting point $(x_0, 0)$ on the positive x–axis; the actual value of x_0 is found by solving $x = (e^{-1/x_0})t + x_0$ for x_0, although this cannot be done explicitly. Whatever the value of x_0 happens to be, the value of u at (x, t) is computed by $u(x, t) = u(x_0, 0) = e^{-1/x_0}$. The solution $u(x, t)$ is then piecewise defined by

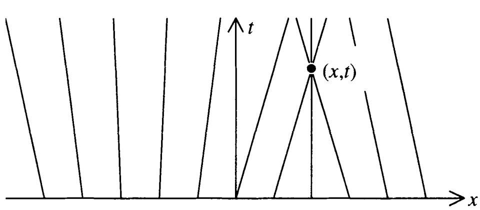
$$u(x,t) = \begin{cases} 0 & \text{if } x \le 0, \\ e^{-1/x_0} & \text{if } x > 0, \text{ where } x_0 + (e^{-1/x_0})t = x. \end{cases}$$

Exercise 17.8. In the last example, find the starting point $(x_0, 0)$ of the characteristic which passes through the point $(x, t) = (1, 2)$. Then calculate the value of the solution u at $(1, 2)$.

Chapter 18

Gradient Catastrophes and Breaking Times

In the previous chapter, it was shown that the solution of a conservation law $u_t + \phi_x = 0$ could be constructed at the point (x,t) by following a characteristic curve from (x,t) back to a point $(x_0, 0)$. An implicit assumption in this method is that there is exactly one characteristic extending from the x-axis to (x,t) in the xt-plane. In nonlinear conservation laws, however, it is possible for two (or more) characteristics to intersect at (x,t):



Such an occurrence can cause the solution $u(x,t)$ to break down with an event called a *gradient catastrophe.* In this chapter we will describe the cause of gradient catastrophes and predict the time at which they occur. As will be discussed in the next chapter, gradient catastrophes are a precursor to shock waves.

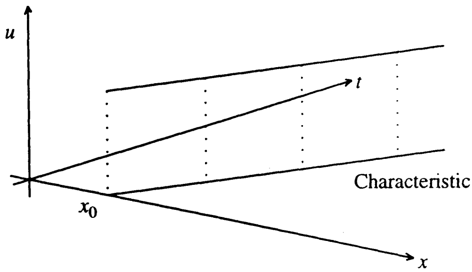


Figure 18.1. Constant value of $u(x, t)$ along a characteristic.

18.1. Gradient catastrophe

In Section 17.4 it was shown that the characteristic curves of the initial value problem

$$(18.1)\qquad \begin{aligned} u_t + c(u)u_x &= 0, \quad -\infty < x < \infty, t > 0, \\ u(x, 0) &= u_0(x) \end{aligned}$$

are lines $x = c(u_0(x_0))t + x_0$ along which the value of u is constant. When viewed in the xtu–diagram shown in Figure 18.1, each characteristic is a line in the xt–plane, and the height of the surface represented by $u(x, t)$ is constant along that line.

In the special case where $c(u)$ is constant (the advection equation), the characteristic lines $x = ct + x_0$ are parallel. By following these characteristics, we see that an initial profile $u(x, 0)$ in the xu–plane has the appearance of being translated along the characteristics as t increases, forming a traveling wave (Figure 18.2).

When $c(u)$ is not constant, however, the characteristic lines $x = c(u_0(x_0))t + x_0$ are not necessarily parallel and may cross. The value of u nevertheless remains constant along each individual characteristic line. As shown in Figure 18.3, if two characteristic lines intersect and the value of u is different along each line, then the slope $u_x(x, t)$ in the x–direction becomes infinite as t approaches the time corresponding to the intersection of the lines. The formation of an infinite slope u_x in the solution u is called a **gradient catastrophe**.

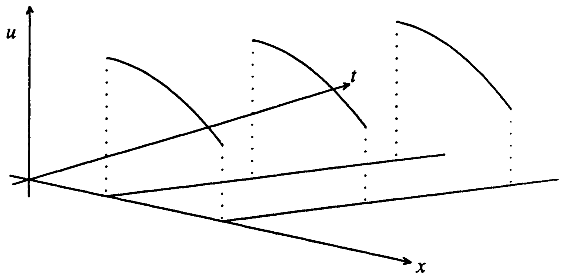


Figure 18.2. Parallel characteristics translate the initial profile in time.

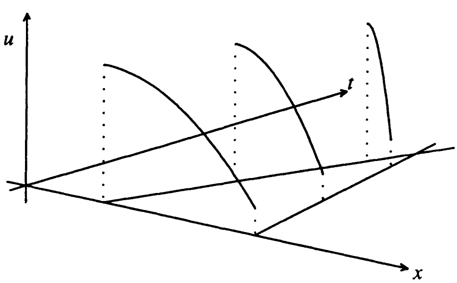


Figure 18.3. Crossing characteristics can result in infinite slope u_x.

The gradient catastrophe can also be seen in the animation of $u(x,t)$. When viewing Figure 18.1 facing the xu–plane, the point $(x(t), t, u(x(t), t))$ on the surface $u = u(x,t)$ above the characteristic curve is projected onto the xu–plane as the point $(x(t), u(x(t), t))$. As t increases, this point appears to move in the xu–plane at a constant height u, since $(x(t), t)$ is following along a characteristic curve. The velocity at which this point moves in the x direction is dx/dt, which by construction of the characteristic curve (17.7) is $dx/dt = c(u(x,t))$. Thus the function $c(u)$ represents the velocity at which a point at height u in the xu–plane animation moves horizontally (Figure 18.4).

Now suppose that $c(u)$ is an increasing function of u, such as $c(u) = u$. In this case, larger values of $u \geq 0$ give larger speeds c, and so the upper part of the profile of $u(x,t)$ (larger values of u) will appear to move to the right faster than the lower part (smaller values

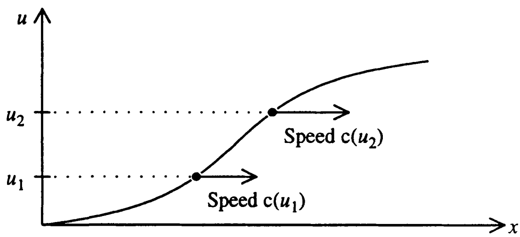


Figure 18.4. Horizontal velocity of a point on the profile of $u(x, t)$ is $c(u)$.

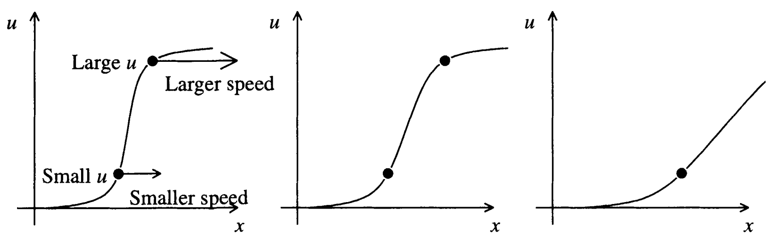


Figure 18.5. Top part of the profile of $u(x, t)$ moves with greater speed than the lower part when $u_t + uu_x = 0$.

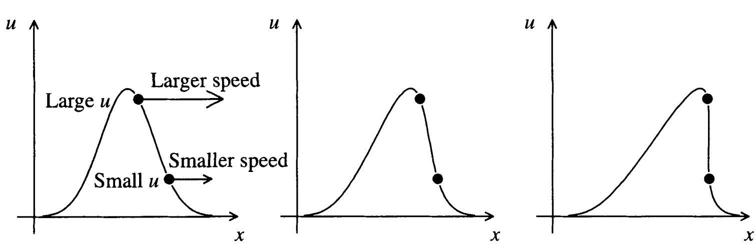


Figure 18.6. Top part of the profile of $u(x, t)$ can catch up to the lower part, forming a gradient catastrophe.

of u). As shown in Figure 18.5, if the profile of $u(x, t)$ at one time is an increasing function of x, then at later times t the profile of $u(x, t)$ will appear to have "thinned out" or rarefied.

On the other hand, if a profile of $u(x, t)$ looks more like a pulse, then the top part of the profile of $u(x, t)$ catches up with the slower moving lower part of the profile (Figure 18.6). This forms an infinite slope u_x, creating a gradient catastrophe. If time were to continue beyond this point, the top part of the profile would appear to overtake the lower part and $u(x,t)$ would fail to be a function.

18.2. Breaking time

The earliest time $t_b \geq 0$ at which a gradient catastrophe occurs in a solution of a conservation law is called the **breaking time**.

Example 18.1. Consider the following initial value problem for the inviscid Burgers equation:

$$u_t + uu_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = e^{-x^2}.$$

With the speed $c(u) = u$ and initial profile $u_0(x) = e^{-x^2}$, the characteristic starting at $(x_0, 0)$ is

$$x = c(u_0(x_0))t + x_0 = e^{-x_0^2}t + x_0.$$

A diagram of characteristics with different starting points $(x_0, 0)$ is displayed in Figure 18.7 and shows that there are characteristics which intersect. From the figure, the earliest time at which characteristics cross appears to be at a breaking time of approximately $t_b = 1.2$.

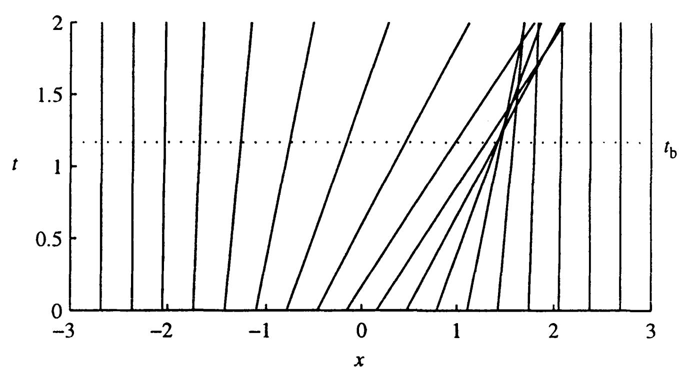


Figure 18.7. Characteristics $x = e^{-x_0^2}t + x_0$ of Example 18.1.

Exercise 18.2. The companion MATLAB software (see page xiii) includes the script `wvburg` for quickly plotting characteristics of the Burgers equation $u_t + uu_x = 0$. In MATLAB, type `wvburg` at the prompt and set the initial profile field $u(x,0)$ to `exp(-x^2)` in order to replicate the characteristics shown in Figure 18.7. By zooming in at the point where characteristics are crossing, verify that the breaking time is about $t_b = 1.2$. What is the breaking time for the solution of the Burgers equation whose initial profile is $u(x,0) = \sin(x)$?

For the remainder of this section, we will discuss how the breaking time t_b can be computed by calculating $u_x(x,t)$ and finding the first time t_b at which u_x becomes infinite.

By the method of characteristics, the value of the solution u of

$$\begin{aligned} u_t + c(u)u_x &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= u_0(x) \end{aligned}$$

at the point (x,t) is $u(x,t) = u_0(x_0)$, where $x_0 = x_0(x,t)$ determines the starting point $(x_0, 0)$ of the characteristic passing through (x,t). The derivative u_x is then

$$u_x(x,t) = u_0'(x_0)\frac{\partial x_0}{\partial x} \tag{18.2}$$

by the chain rule.

The value of x_0 which determines the starting point $(x_0, 0)$ of the characteristic through (x,t) is defined implicitly by the equation

$$x = c(u_0(x_0))t + x_0.$$

The derivative of x_0 with respect to x can be found from this equation by implicit differentiation. Taking the partial derivative of both sides with respect to x gives

$$\begin{aligned} \frac{\partial}{\partial x}x &= \frac{\partial}{\partial x}\Big[c(u_0(x_0))t + x_0\Big], \\ 1 &= t\frac{d}{dx_0}\Big[c(u_0(x_0))\Big]\frac{\partial x_0}{\partial x} + \frac{\partial x_0}{\partial x}. \end{aligned}$$

Solving for $\partial x_0/\partial x$ then shows that

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + t\,\frac{d}{dx_0}c(u_0(x_0))}.$$

Substituting this into (18.2) expresses the derivative of $u(x,t)$ with respect to x as

$$u_x(x,t) = \frac{u_0'(x_0)}{1 + t\,\frac{d}{dx_0}c(u_0(x_0))}. \tag{18.3}$$

The problem of determining when u_x becomes infinite is now reduced to a problem of determining when the denominator of (18.3) approaches zero.

If $\frac{d}{dx_0}c(u_0(x_0)) \geq 0$ for all initial points $(x_0, 0)$, then the denominator in (18.3) never approaches 0 as t increases from zero. In this case, no gradient catastrophe occurs. On the other hand, if $\frac{d}{dx_0}c(u_0(x_0))$ is negative for some x_0, then a gradient catastrophe can occur since the denominator in (18.3) will approach 0 as t approaches $-1/\frac{d}{dx_0}c(u_0(x_0))$. The value of x_0 which produces the earliest blowup time t is the value of x_0 which makes $\frac{d}{dx_0}c(u_0(x_0))$ the most negative. Using this value of x_0, the breaking time is then

$$\boxed{t_b = \frac{-1}{\frac{d}{dx_0}c(u_0(x_0))}.} \tag{18.4}$$

Example 18.3. Returning to the initial value problem in Example 18.1, the expression (18.4) will be used to compute the breaking time t_b in Figure 18.7. With the speed $c(u) = u$ and initial profile $u_0(x) = e^{-x^2}$ from that example, the speed of the characteristic starting at $(x_0, 0)$ is

$$c(u_0(x_0)) = c(e^{-x_0^2}) = e^{-x_0^2}.$$

The breaking time t_b in (18.4) requires finding the most negative value of

$$F(x_0) = \frac{d}{dx_0}c(u_0(x_0)) = \frac{d}{dx_0}e^{-x_0^2} = -2x_0e^{-x_0^2}.$$

The derivative $F'(x_0) = (-2+4x_0^2)e^{-x_0^2}$ shows that $F(x_0)$ has critical points at $x_0 = \pm 1/\sqrt{2}$, with $x_0 = 1/\sqrt{2}$ yielding the most negative value of $F(x_0)$. The breaking time (18.4) with $x_0 = 1/\sqrt{2}$ is then

$$t_b = \frac{-1}{-2x_0e^{-x_0^2}} = \frac{1}{\sqrt{2}e^{-1/2}} = \sqrt{\frac{e}{2}}.$$

The value of $t_b = \sqrt{e/2}$ is approximately 1.16 and is shown earlier in Figure 18.7.

Exercise 18.4. Consider the initial value problem

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = \frac{1}{1+x^2}.$$

(a) Find the equations of the characteristics for this problem. Using a computer or graphing calculator, graph several characteristics on the same screen (such as Figure 18.7) to identify where a gradient catastrophe might occur. Based on the diagram, estimate a value of the breaking time t_b.

(b) Calculate the breaking time t_b by using (18.4) to determine the first time that u_x becomes infinite.

Chapter 19

Shock Waves

The derivation of the differential equation form of a conservation law $u_t + \phi_x = 0$ assumes that the solution u has continuous first derivatives. The method of characteristics can construct such a solution, but only up until the time of a gradient catastrophe. In this chapter the solution $u(x,t)$ will be extended beyond the breaking time by permitting $u(x,t)$ to be a *piecewise smooth function.* In doing so, we will have to return to the original integral form of the conservation law at points (x,t) where $u(x,t)$ is discontinuous. The resulting discontinuous solution of the conservation law is called a *shock wave.*

19.1. Piecewise smooth solutions of a conservation law

As we have seen, characteristic curves for the initial value problem

$$u_t + c(u)u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = u_0(x)$$

can be used to construct a solution $u(x,t)$ starting at time $t = 0$, but ending at the breaking time t_b of a gradient catastrophe. In the following section we will modify the method of characteristics to allow the profile $u(x,t)$ to literally break at time $t = t_b$, forming a function which is only *piecewise smooth* for time $t \geq t_b$ (Figure 19.1).

To describe piecewise smooth functions, suppose $(x_s(t), t)$ is a curve in the xt-plane which divides the upper half of the plane into two parts (Figure 19.2). Let R^- represent the region to the left of the

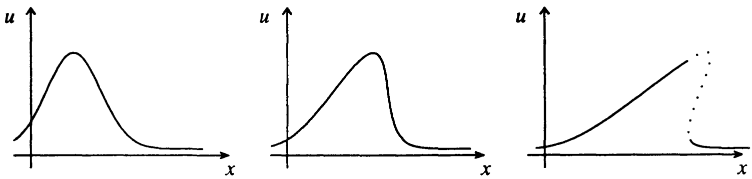


Figure 19.1. Profiles of a function $u(x,t)$ which "breaks" after a gradient catastrophe.

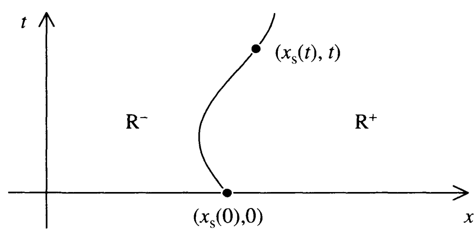


Figure 19.2

curve and R^+ the region to the right of the curve. A function $u(x,t)$ is called a **piecewise smooth solution** of

$$u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = u_0(x)$$

with **jump discontinuity** along x_s if $u(x,t)$ has the following properties:

(1) $u(x,t)$ has continuous first derivatives u_t and u_x in R^+ and R^-, and satisfies the initial value problem in region R^-

$$u_t + \phi_x = 0 \qquad \text{for } (x,t) \text{ in } R^-,$$
$$u(x,0) = u_0(x) \quad \text{for } x < x_s(0),$$

and in region R^+

$$u_t + \phi_x = 0 \qquad \text{for } (x,t) \text{ in } R^+,$$
$$u(x,0) = u_0(x) \quad \text{for } x > x_s(0).$$

(2) At each point (x_0, t_0) on the curve $(x_s(t), t)$, the limit of $u(x,t)$ as $(x,t) \to (x_0,t_0)$ in R^- and the limit of $u(x,t)$ as

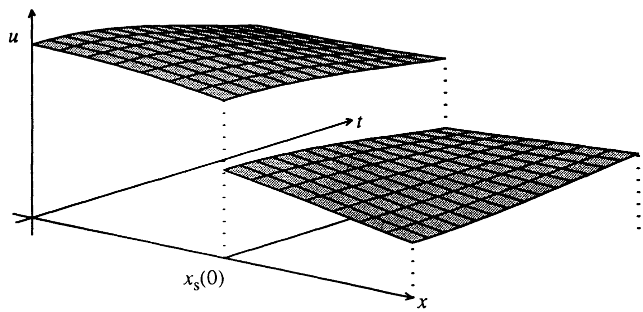


Figure 19.3. The graph of a piecewise smooth function $u(x, t)$.

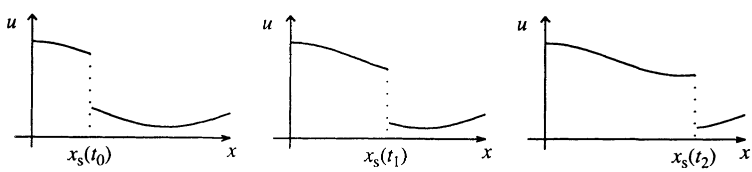


Figure 19.4. Profiles of a piecewise smooth function $u(x, t)$ with discontinuity at $x_s(t)$.

$(x, t) \to (x_0, t_0)$ in R^+ both exist but are not necessarily equal.

The graph of such a function appears as two sections of surface with a jump along the curve $(x_s(t), t)$ in the xt−plane (Figure 19.3). The animation of a piecewise smooth function, formed by taking slices of the surface at a sequence of increasing times, has a profile with a moving jump discontinuity located at $x_s(t)$ (Figure 19.4).

19.2. Shock wave solutions of a conservation law

The construction of a solution of $u_t + \phi_x = 0$ by the method of characteristics temporarily stops when a gradient catastrophe occurs. The physical process that the conservation law models, however, does not necessarily end. In this section we will describe how to extend the solution $u(x, t)$ beyond the breaking time by permitting $u(x, t)$ to be only piecewise smooth, but in a way which continues to obey the

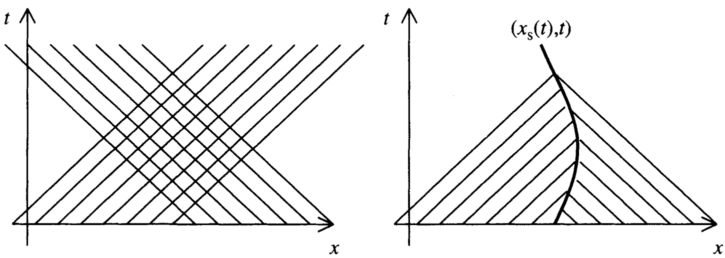


Figure 19.5. Using a curve to divide a region of crossing characteristics.

underlying conservation principle. The formation of a discontinuity after a gradient catastrophe is a dramatic change in the nature of $u(x,t)$. Such a function will be called a *shock wave* solution of the conservation law.

Suppose that characteristics of

$$(19.1)\qquad \begin{aligned} &u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = u_0(x) \end{aligned}$$

begin intersecting at time t_b, which we will assume is $t_b = 0$ as shown in Figure 19.5. In order to proceed with the method of characteristics, a curve $(x_s(t), t)$ is drawn through the region of crossing characteristics to separate the characteristics approaching from the left and right (Figure 19.5). While many curves can be drawn to separate the crossing characteristics, it will now be shown that the underlying conservation law selects out one choice of $x_s(t)$.

Suppose $u(x,t)$ is a piecewise smooth solution of the initial value problem (19.1) with jump discontinuity along $x_s(t)$. While $u(x,t)$ satisfies $u_t + \phi_x = 0$ at each point (x,t) in R^- and R^+, the derivatives of $u(x,t)$ do not necessarily exist at points (x,t) on the curve. To see what happens at points $(x_s(t),t)$ on the curve, we have to return to the original integral form of the conservation law (15.5). With no source term, the integral form of the conservation law is

$$(19.2)\qquad \frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t).$$

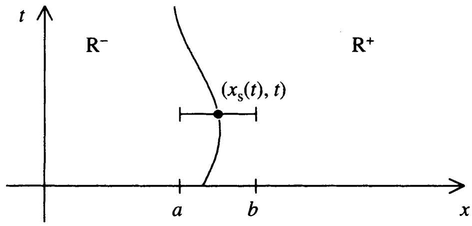


Figure 19.6

Fixing a point $(x_s(t), t)$ on the curve, pick a and b so that $a < x_s(t) < b$ as shown in Figure 19.6. The integral in the conservation law (19.2) can then be split into two parts as

$$\int_a^b u(x,t)dx = \int_a^{x_s(t)^-} u(x,t)dx + \int_{x_s(t)^+}^b u(x,t)dx.$$

Substituting into the conservation law (19.2) and using the chain rule to compute the derivative of these integrals with respect to t results in

$$\int_a^{x_s(t)^-} u_t(x,t)dx + u(x_s^-,t)\frac{dx_s}{dt} + \int_{x_s(t)^+}^b u_t(x,t)dx - u(x_s^+,t)\frac{dx_s}{dt} = \phi(a,t) - \phi(b,t).$$

Letting $a \to x_s^-$ and $b \to x_s^+$ reduces this to the equation

$$u(x_s^-,t)\frac{dx_s}{dt} - u(x_s^+,t)\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t),$$

from which we can solve for dx_s/dt to obtain the ordinary differential equation

$$\frac{dx_s}{dt} = \frac{\phi(x_s^+,t) - \phi(x_s^-,t)}{u(x_s^+,t) - u(x_s^-,t)}. \tag{19.3}$$

This derivation shows that in order for a piecewise smooth solution of the initial value problem (19.1) to satisfy the integral form of the conservation law (19.2), the curve along which $u(x,t)$ has a jump discontinuity must be picked to satisfy (19.3). The differential equation (19.3) is called the **Rankine-Hugoniot jump condition**

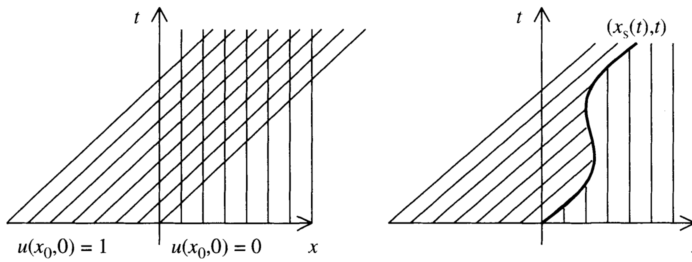


Figure 19.7. Characteristic diagram and a curve separating crossing characteristics for Example 19.1.

for $u(x,t)$. The expressions $\phi(x_s^+,t) - \phi(x_s^-,t)$ and $u(x_s^+,t) - u(x_s^-,t)$ calculate the jump in the values of ϕ and u as (x,t) crosses the curve $(x_s(t),t)$ from left to right. Using the jump notation

$$[\phi](x,t) = \phi(x^+,t) - \phi(x^-,t), \quad [u](x,t) = u(x^+,t) - u(x^-,t),$$

the Rankine-Hugoniot jump condition is written as

$$\boxed{\frac{dx_s}{dt} = \frac{[\phi]}{[u]}.}$$

A piecewise smooth solution $u(x,t)$ of $u_t + \phi_x = 0$ with a jump along a curve $x_s(t)$ satisfying the Rankine-Hugoniot condition is called a **shock wave** solution of the conservation law. The curve $x_s(t)$ is called a **shock path**.

Example 19.1. Consider the following initial value problem for the inviscid Burgers equation:

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The characteristics $x = c(u_0(x_0))t + x_0$ of this problem are $x = 0 \cdot t + x_0$ when $x_0 > 0$, and $x = 1 \cdot t + x_0$ when $x_0 < 0$. Based on the diagram of characteristics (Figure 19.7), it appears that the characteristics begin crossing at $(0,0)$ with a breaking time of $t_b = 0$. For this reason we will look for a shock wave solution with shock path starting at $(0,0)$.

Once $(x_s(t), t)$ is found to separate the crossing characteristics, the method of characteristics can be used in the regions R^- to the left and R^+ to the right of the path (Figure 19.7). If (x, t) is a point in R^-, then there is one characteristic line extending back from (x, t) to a point $(x_0, 0)$ on the negative x−axis. Since u is constant along this line and the value of $u(x_0, 0) = 1$ for $x_0 < 0$, the value of u at (x, t) is $u(x, t) = u(x_0, 0) = 1$. Similarly, if (x, t) is a point in R^+, then the characteristic through it extends back to a point $(x_0, 0)$ on the positive x−axis where $u(x_0, 0) = 0$. In this case, $u(x, t) = u(x_0, 0) = 0$. Once the shock path is found to separate the regions R^- and R^+, the solution u will be given by

$$u(x,t) = \begin{cases} 1 & \text{if } (x,t) \in R^-, \\ 0 & \text{if } (x,t) \in R^+. \end{cases}$$

The curve $(x_s(t), t)$ separating the two regions will be found using the Rankine-Hugoniot jump condition; starting the shock path at $(0, 0)$ forms the initial value problem

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}, \quad x_s(0) = 0.$$

The flux ϕ for the Burgers equation $u_t + uu_x = 0$ is $\phi = \frac{1}{2}u^2$, so

$$\frac{dx_s}{dt} = \frac{[\frac{1}{2}u^2]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}.$$

Since $u = 1$ in R^- and $u = 0$ in R^+, the value of u as (x, t) approaches the curve from the left is $u^- = 1$, while the value from the right is $u^+ = 0$. The jump condition then simplifies to $dx_s/dt = 1/2$, which together with the initial condition $x_s(0) = 0$ implies the shock path is the line $x_s = t/2$. An xt−diagram showing the shock path $x = t/2$ and characteristics (Figure 19.8) illustrates the resulting shock wave solution,

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}t, \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases}$$

Four frames of animation of this function are shown in Figure 19.9. Note in particular the jump discontinuity moving to the right with speed $1/2$.

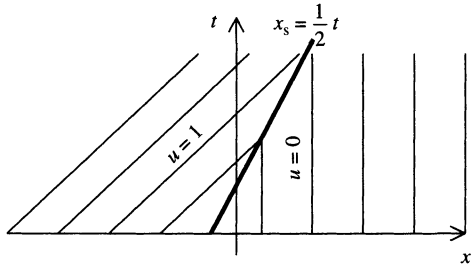


Figure 19.8. Shock path for Example 19.1.

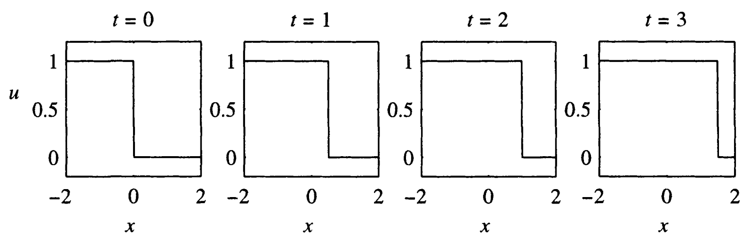


Figure 19.9. Animation of the shock wave solution of Example 19.1.

Exercise 19.2. Find a shock wave solution for the following initial value problem, and then animate the result:

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 2 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Chapter 20

Shock Wave Example: Traffic at a Red Light

Shock wave solutions for conservation laws are piecewise smooth solutions which satisfy the Rankine-Hugoniot jump condition along curves of discontinuity. The resulting moving discontinuity models an abrupt change propagating through a medium. In this chapter a shock wave will be constructed to model traffic backing up at a red light.

20.1. An initial value problem

Suppose that car traffic, moving uniformly along a single lane road, encounters the end of a line of traffic which has stopped at a traffic light (Figure 20.1). The cars which have already stopped are lined up with maximum density u_1 cars per mile, while the cars approaching the end of the line have a uniform density u_0 cars per mile. Since u_1 is the maximum possible traffic density, the value of u_0 will satisfy $0 < u_0 < u_1$.

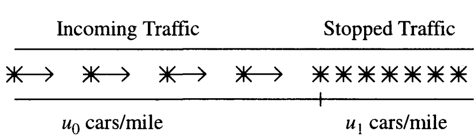


Figure 20.1. Incoming cars encountering stopped traffic.

Returning to Section 16.3, let $u(x,t)$ represent the density (cars per mile) of traffic at position x along the road at time t. The flux $\phi(x,t)$ represents the rate (cars per hour) at which traffic passes by position x and time t. Letting v_1 denote maximum traffic velocity, the linear model for traffic velocity $v = v_1(1 - u/u_1)$ results in the constitutive equation (see Section 16.3)

$$\phi = uv = v_1(u - u^2/u_1). \tag{20.1}$$

Assuming that the road has no entrances or exits, the basic conservation law $u_t + \phi_x = f$ with flux ϕ and source $f = 0$ becomes

$$u_t + v_1(1 - 2u/u_1)u_x = 0.$$

Let $x = 0$ represent the location of the end of the stopped traffic at time $t = 0$. For now, it will be assumed that the stopped traffic extends indefinitely in one direction and the incoming traffic extends indefinitely in the other. In this case, the initial value problem

$$\begin{aligned} &u_t + v_1(1 - 2u/u_1)u_x = 0, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \begin{cases} u_0 & \text{if } x < 0, \\ u_1 & \text{if } x \geq 0, \end{cases} \end{aligned} \tag{20.2}$$

models the profile of traffic density $u(x,t)$ at later times t.

20.2. Shock wave solution

In this section we will use the method of characteristics to find a solution of the initial value problem (20.2). Since the conservation law in (20.2) is of the form $u_t + c(u)u_x = 0$, a solution u of (20.2) will be constant along the characteristic lines

$$x = c(u(x_0, 0))t + x_0,$$

where $u(x_0, 0)$ is determined by the initial condition in (20.2), and $c(u)$ is given by

$$c(u) = v_1(1 - 2u/u_1).$$

If $x_0 \geq 0$, then the characteristic starting at $(x_0, 0)$ is

$$x = c(u_1)t + x_0 = -v_1 t + x_0.$$

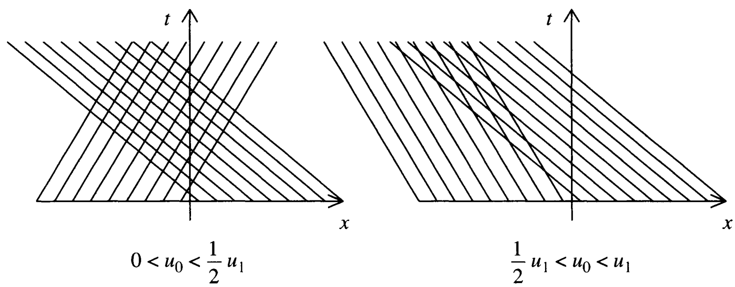


Figure 20.2. Breaking time for the solution of the initial value problem (20.2) is $t_b = 0$.

In an xt–diagram, this shows that characteristics starting at points $(x_0, 0)$ on the positive x–axis are parallel lines with negative slope $-1/v_1$.

On the other hand, if $x_0 < 0$, then the characteristic starting at $(x_0, 0)$ is

$$x = c(u_0)t + x_0 = v_1(1 - 2u_0/u_1)t + x_0.$$

Note that this line can have positive or negative slope $1/c$ depending on whether $c = v_1(1 - 2u_0/u_1)$ is positive or negative, i.e., if u_0 is smaller or larger than $u_1/2$ (see Figure 20.2). In either case, however, the slope $1/c$ will be between $-1/v_1$ and $1/v_1$ since the incoming traffic density u_0 satisfies $0 < u_0 < u_1$.

As shown in Figure 20.2, the characteristics will begin crossing at the origin. For this problem we will need to look for a shock wave solution whose shock path $x_s(t)$ starts at $x_s(0) = 0$ and extends upward to divide the region in which characteristics intersect (Figure 20.3).

At a point (x, t) to the left of the shock path, the characteristic passing through the point extends back to the negative x–axis where $u(x_0, 0) = u_0$. Since u is constant along characteristics, $u(x, t) = u_0$. Similarly, a point (x, t) to the right of the shock path lies on a characteristic which extends back to the positive x–axis where $u(x_0, 0) = u_1$, so $u(x, t) = u_1$. The traffic density function $u(x, t)$ will then have

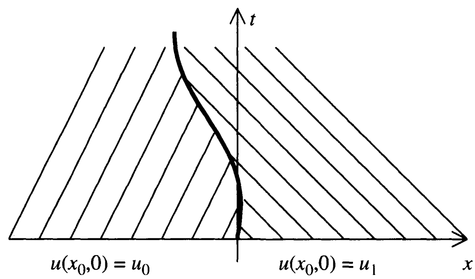


Figure 20.3. Setting up an xt–diagram for the initial value problem (20.2).

the form

$$u(x,t) = \begin{cases} u_0 & \text{if } x < x_s(t), \\ u_1 & \text{if } x > x_s(t). \end{cases}$$

The Rankine-Hugoniot jump condition $dx_s/dt = [\phi]/[u]$ will determine the shock path with the flux ϕ given by (20.1). At a point (x,t) on the shock path, we already determined that the values of u from the right and left are $u^+ = u_1$ and $u^- = u_0$. The jump condition

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\phi(u^+) - \phi(u^-)}{u^+ - u^-} = \frac{\phi(u_1) - \phi(u_0)}{u_1 - u_0}$$

then simplifies to

$$\frac{dx_s}{dt} = \frac{0 - v_1(u_0 - u_0^2/u_1)}{u_1 - u_0} = -v_1\frac{u_0}{u_1}.$$

Integrating this differential equation with respect to t and using the starting point $x_s(0) = 0$ gives the only allowed shock path, the line

$$x_s = -v_1\frac{u_0}{u_1}t.$$

The resulting shock wave solution to (20.2) is then

$$u(x,t) = \begin{cases} u_0 & \text{if } x < -v_1(u_0/u_1)t, \\ u_1 & \text{if } x \geq -v_1(u_0/u_1)t, \end{cases} \tag{20.3}$$

with an xt–diagram shown in Figure 20.4. Note that this shock path indicates that the end of the line of stopped traffic will back up at the rate of $v_1(u_0/u_1)$ miles per hour.

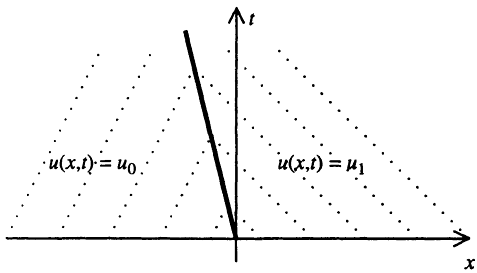


Figure 20.4. The shock path $x_s = -v_1(u_0/u_1)t$.

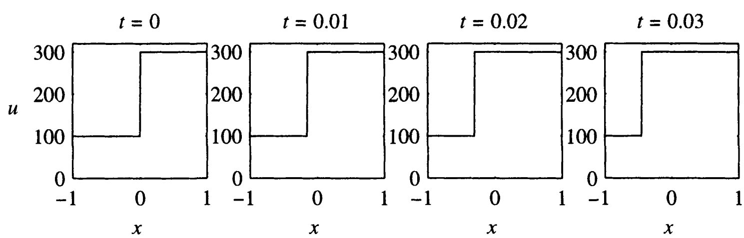


Figure 20.5. Traffic backing up at a rate of 15 miles per hour.

Example 20.1. As a particular example, suppose that the stopped traffic is at a maximum density $u_1 = 300$ cars per mile, and the maximum velocity along this stretch of road is $v_1 = 45$ miles per hour. If the incoming traffic is traveling at 30 miles per hour, then the velocity model $v = v_1(1-u/u_1)$ predicts that the incoming traffic density u_0 satisfies $30 = 45(1 - u_0/300)$, so $u_0 = 100$ cars per mile. With these values, the solution (20.3) becomes

$$u(x,t) = \begin{cases} 100 & \text{if } x < -15t, \\ 300 & \text{if } x \geq -15t. \end{cases}$$

The resulting shock path, representing the location of the end of the line of stopped traffic, is given by $x = -v_1(u_0/u_1)t = -15t$, indicating that the end of the stopped traffic is backing up at 15 miles per hour. Four frames of animation of this traffic flow are shown in Figure 20.5.

Exercise 20.2. Suppose that the incoming traffic in Example 20.1 is traveling at 15 miles per hour.

(a) Consider the speed at which the end of the line of stopped traffic propagates backwards through the incoming traffic. Without computing the shock path, would you say that this speed should be larger or smaller than the rate 15 miles per hour in Example 20.1?

(b) Find the shock path and resulting shock wave solution of the initial value problem (20.2). Then represent the solution u with an xt–diagram. How is the speed of the shock path here different from the one in Example 20.1?

Exercise 20.3. Suppose car velocity is modeled by $v = v_1(1 - u^2/u_1^2)$.

(a) Sketch the graph of v as a function of the traffic density u. In what ways is this model of car velocities more (or less) realistic than the linear model $v = v_1(1 - u/u_1)$?

(b) Find a shock wave solution of

$$u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} u_0 & \text{if } x < 0, \\ u_1 & \text{if } x \geq 0, \end{cases}$$

with the flux modeled by $\phi = uv = v_1(u - u^3/u_1^2)$. Sketch the resulting xt–diagram of the solution.

Chapter 21

Shock Waves and the Viscosity Method

Another approach for locating shock waves is through the use of *viscosity solutions.* By making a modification to the model of traffic stopping at a red light, we will construct viscosity solutions which resemble the shock wave found in the previous chapter. As the amount of viscosity is decreased, the viscosity solution approaches the shock wave solution.

21.1. Another model of traffic flow

The traffic flow in the previous chapter was described by a conservation law $u_t + \phi_x = 0$ and flux $\phi = uv$, where $v = v_1(1 - u/u_1)$ modeled the traffic velocity v that occurs when traffic is at a density u. Changes to this model can be made to take into account other factors which may influence traffic velocity. Here we will modify the model to incorporate a driver's ability to look ahead and sense upcoming changes in traffic density.

As shown in Figure 21.1, if traffic is moving in the positive x direction, then the sign of $u_x(x, t)$ at time t indicates whether the traffic ahead of position x is more dense ($u_x > 0$) or less dense ($u_x < 0$) than the traffic at position x. Suppose that drivers will adjust their speed downward if they see an increase in traffic density ahead

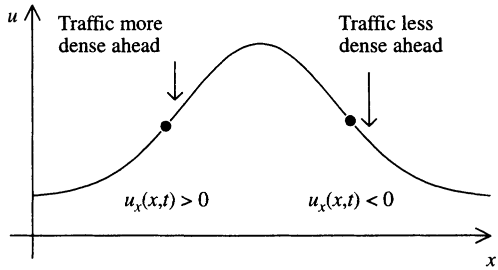


Figure 21.1

of them, and adjust their speed upward if they see a decrease in traffic density. One model, which uses the derivative u_x to account for this adjustment of traffic velocity, is

$$v = v_1(1 - u/u_1) - ru_x/u.$$

This equation starts with the basic linear model of traffic velocity given by $v = v_1(1 - u/u_1)$, and then makes an adjustment to the velocity by an amount $-ru_x/u$. The fraction u_x/u is a measure of the relative change in the traffic density, and the constant r is a positive number which indicates how sensitive drivers are to changes in traffic density.

With this model of traffic velocity, the flux becomes

$$\phi = uv = v_1(u - u^2/u_1) - ru_x,$$

and so the resulting conservation law $u_t + \phi_x = 0$ is now the second order equation

$$\boxed{u_t + v_1(1 - 2u/u_1)u_x = ru_{xx}.} \tag{21.1}$$

When $r = 0$, this equation reduces to the traffic flow model used in the previous chapter.

In the spirit of uniform traffic reaching the end of a line of stopped traffic, we will assume that far ahead on the road the traffic density is nearly constant and at its maximum density. This will be represented in terms of limits by

$$\lim_{x\to\infty} u(x,0) = u_1, \quad \lim_{x\to\infty} u_x(x,0) = 0. \tag{21.2a}$$

Much further back on the road, the incoming traffic density is assumed to be nearly constant at u_0 cars per mile, expressed in terms of limits as

$$\lim_{x \to -\infty} u(x,0) = u_0, \quad \lim_{x \to -\infty} u_x(x,0) = 0. \tag{21.2b}$$

Exercise 21.1. Let $U = v_1(1 - 2u/u_1)$. Show that this substitution in (21.1) results in the *viscous* Burgers equation $U_t + UU_x = rU_{xx}$.

21.2. Traveling wave solutions of the new model

In this section we will find traveling wave solutions of the modified traffic flow model

$$u_t + v_1(1 - 2u/u_1)u_x = ru_{xx}.$$

Making the substitution $u(x,t) = f(x - ct)$ in this model produces the ordinary differential equation

$$-cf' + v_1 f' - \frac{2v_1}{u_1} ff' = rf''.$$

Integrating once then results in

$$-cf + v_1 f - \frac{v_1}{u_1} f^2 = rf' + k \tag{21.3}$$

for some constant of integration k.

At this point the constant k and speed c can be found by using the limit assumptions in (21.2). Since $u(x,0) = f(x)$, the limiting conditions (21.2) show that f should satisfy

$$\lim_{z \to \infty} f(z) = u_1, \qquad \lim_{z \to -\infty} f(z) = u_0,$$
$$\lim_{z \to \infty} f'(z) = 0, \qquad \lim_{z \to -\infty} f'(z) = 0.$$

Taking $z \to \infty$ and $z \to -\infty$ in the differential equation (21.3) results in the two equations

$$-cu_1 + v_1 u_1 - v_1 u_1 = r \cdot 0 + k,$$
$$-cu_0 + v_1 u_0 - \frac{v_1 u_0^2}{u_1} = r \cdot 0 + k.$$

Solving these two algebraic equations for k and c gives the values

$$c = -\frac{u_0 v_1}{u_1}, \qquad k = v_1 u_0. \tag{21.4}$$

With these values of c and k, the differential equation in (21.3) then becomes

$$\frac{u_0 v_1}{u_1} f + v_1 f - \frac{v_1}{u_1} f^2 = r f' + v_1 u_0,$$

which after rearranging can be written as

$$(f - u_0)(f - u_1) = -\frac{r u_1}{v_1} f'.$$

Separating variables by writing

$$\frac{f'}{(f - u_0)(f - u_1)} = -\frac{v_1}{r u_1}$$

and integrating both sides with respect to z (using partial fractions on the left side) gives the general solution

$$\frac{1}{u_1 - u_0} \ln \left| \frac{f - u_1}{f - u_0} \right| = -\frac{v_1}{r u_1} z + k_1.$$

Since $u(x,t) = f(x - ct)$ represents the traffic density as cars with density u_0 approach a line of traffic of maximum density u_1, we will assume that the value of $f(x - ct)$ is between u_0 and u_1, that is, $u_0 < f(z) < u_1$. In this case

$$\frac{1}{u_1 - u_0} \ln \left[\frac{u_1 - f}{f - u_0} \right] = -\frac{v_1}{r u_1} z + k_1.$$

Taking $k_1 = 0$ and solving for $f(z)$ gives the traveling wave profile

$$f(z) = \frac{u_1 + u_0 \exp\left[-\frac{(u_1 - u_0) v_1}{r u_1} z \right]}{1 + \exp\left[-\frac{(u_1 - u_0) v_1}{r u_1} z \right]}.$$

Adding and subtracting the term $u_1 \exp[\ \cdot\ ]$ in the numerator and splitting the result into two fractions shows that $f(z)$ can also be written in the form

$$f(z) = u_1 + \frac{u_0 - u_1}{1 + \exp\left[\frac{v_1 (u_1 - u_0)}{r u_1} z \right]}.$$

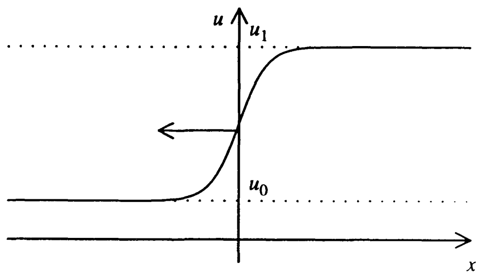


Figure 21.2. A profile of the traveling wave solution (21.5) of the modified traffic flow equation (21.1).

The traveling wave solution $u(x,t) = f(x - ct)$ with c given by (21.4) is then

$$u(x,t) = u_1 + \frac{u_0 - u_1}{1 + \exp\left[\frac{v_1(u_1 - u_0)}{ru_1}\left(x + \frac{v_1 u_0}{u_1}\, t\right)\right]}. \tag{21.5}$$

A profile of this wave is shown in Figure 21.2.

21.3. Viscosity

The term ru_{xx} added to form the traffic flow equation

$$u_t + v_1(1 - 2u/u_1)u_x = ru_{xx}$$

is called a **viscosity** term with **viscosity parameter** r. The resulting solution (21.5) is called a **viscosity solution**. This solution is similar to the shock wave solution (20.3) found in Section 20.2: both are traveling waves moving against the flow of incoming traffic with speed $c = v_1 u_0/u_1$. Their profile shapes are similar, too, except that the viscosity solution does not have a discontinuity (Figure 21.2). The effect of adding the viscosity term to form the modified equation $u_t + c(u)u_x = ru_{xx}$ is to "smooth out" the discontinuous shock wave solution of $u_t + c(u)u_x = 0$.

As r decreases to 0, the viscosity solution $u(x,t)$ given by (21.5) approaches the traveling shock wave. If $x + (v_1 u_0/u_1)t > 0$, then

$$\lim_{r \to 0^+} u(x,t) = \lim_{r \to 0^+} \left[u_1 + \frac{u_0 - u_1}{1 + \exp\left[\frac{v_1(u_1-u_0)}{ru_1}(x + \frac{v_1 u_0}{u_1} t)\right]} \right]$$
$$= u_1,$$

since the exponential grows to infinity. On the other hand, if $x + (v_1 u_0/u_1)t < 0$, then the exponential decays to zero and

$$\lim_{r \to 0^+} u(x,t) = \lim_{r \to 0^+} \left[u_1 + \frac{u_0 - u_1}{1 + \exp\left[\frac{v_1(u_1-u_0)}{ru_1}(x + \frac{v_1 u_0}{u_1} t)\right]} \right]$$
$$= u_1 + (u_0 - u_1) = u_0.$$

Thus as the viscosity parameter r decreases to zero, the viscosity solution approaches the shock wave solution (20.3) given by

$$\lim_{r \to 0^+} u(x,t) = \begin{cases} u_0 & \text{if } x < -v_1(u_0/u_1)t, \\ u_1 & \text{if } x > -v_1(u_0/u_1)t. \end{cases}$$

This approach to finding the shock solution is called the *viscosity method* for constructing shock waves.

Exercise 21.2. Suppose the maximum velocity v_1 is 60 miles per hour, the maximum density u_1 is 350 cars per mile, and the incoming traffic density u_0 is 100 cars per mile. Animate the viscosity solution (21.5) with $r = 100$, 10, 1, 0.1, and 0.01.

Chapter 22

Rarefaction Waves

Earlier we saw how intersecting characteristics led to the construction of shock wave solutions of a conservation law. In this chapter we will examine a problem at the other extreme: in nonlinear conservation laws, it is possible to have regions in the xt-plane which contain no characteristics. For these regions, the method of characteristics will be modified to form *rarefaction waves*. Later in this chapter a rarefaction wave will be constructed which models traffic flow after a red light turns green.

22.1. An example of a rarefaction wave

The characteristics $x = c(u(x_0, 0))t + x_0$ for the initial value problem

$$u_t + uu_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{22.1}$$

constructed using the characteristic speed $c(u) = u$ are

$$x = 0 \cdot t + x_0 \quad \text{if } x_0 \leq 0,$$
$$x = 1 \cdot t + x_0 \quad \text{if } x_0 > 0.$$

When drawn in the xt-plane (Figure 22.1), note that the characteristics do not enter the wedge-shaped region $0 < x < t < \infty$. In this section we will look at *rarefaction waves* as one way of constructing a solution $u(x,t)$ of the initial value problem (22.1) in this region.

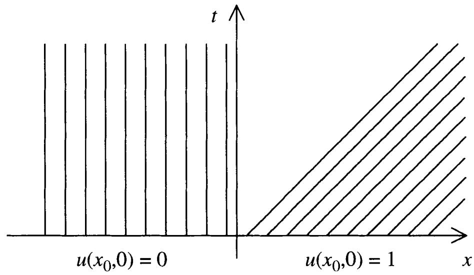


Figure 22.1. Characteristics which do not enter part of the xt–plane.

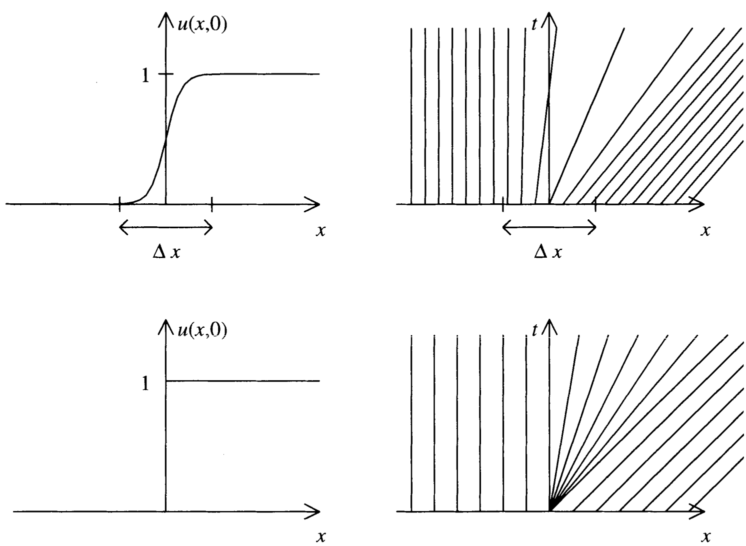


Figure 22.2. Smoothing the initial data $u(x,0)$ to create a fan of characteristics, then letting $\Delta x \to 0$.

Suppose the initial profile $u(x,0)$ is modified to make a smooth transition from $u = 0$ to $u = 1$ within an interval of length Δx around $x = 0$. As shown in Figure 22.2, the resulting characteristics then make a smooth transition from lines with speed $c = 0$ (vertical) to lines with speed $c = 1$ (slope 1). Letting the interval of transition Δx shrink to 0 (Figure 22.2) suggests that we might be able to find a solution of $u_t + uu_x = 0$ in the region $0 < x < t$ by filling it with a "fan of characteristics". This fan consists of lines $x = ct$, originating from the origin, whose speeds vary from $c = 0$ (vertical line) to $c = 1$. A function $u(x,t)$ which is constant along each of these inserted "characteristics" would be of the form $u(x,t) = g(x/t)$, a function of the speed (or slope) of the lines $x = ct$.

To search for a solution of $u_t + uu_x = 0$ of the form $u(x,t) = g(x/t)$, first note that by the chain rule, the derivatives u_t and u_x are

$$u_t(x,t) = -\frac{x}{t^2}g'(x/t), \quad u_x(x,t) = \frac{1}{t}g'(x/t).$$

Substituting these derivatives into $u_t + uu_x = 0$ produces the equation

$$-\frac{x}{t^2}g'(x/t) + g(x/t) \cdot \frac{1}{t}\, g'(x/t) = 0,$$

from which it follows by factoring that

$$\frac{1}{t}g'(x/t)\left(g(x/t) - \frac{x}{t}\right) = 0.$$

This shows that either $g' = 0$ (g is constant) or $g(x/t) = x/t$. The following exercise shows that we can discard the first possibility.

Exercise 22.1. Consider the initial value problem given in (22.1). Use the method of characteristics to show that $u(x,t) = 0$ in the region $x \leq 0$ and $u(x,t) = 1$ in the region $x > t$. Now suppose that $u(x,t) = g(x/t) = A$ in the wedge-shaped region $0 < x < t$, resulting in the function

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0, \\ A & \text{if } 0 < x \leq t, \\ 1 & \text{if } t < x. \end{cases}$$

Use the Rankine-Hugoniot jump condition along the lines $x = 0$ and $x = t$ to show that $u(x,t)$ cannot be a shock wave solution of (22.1).

The other possibility for g is $g(x/t) = x/t$. Figure 22.3 shows the resulting xt–diagram that is formed by taking $u(x,t) = g(x/t) = x/t$ in the wedge-shaped region $0 < x < t$, and using the method of characteristics in the left ($x < 0$) and right ($x > t$) regions. The

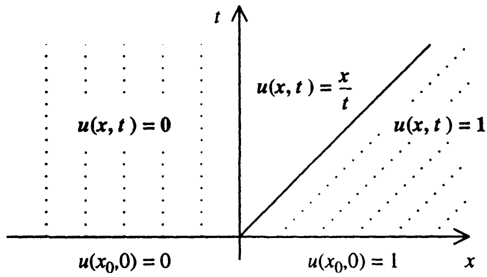


Figure 22.3. An xt–diagram using $u(x,t) = x/t$ to fill the center wedge-shaped region.

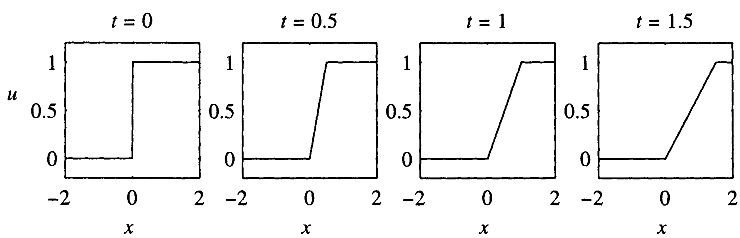


Figure 22.4. Animation of the function $u(x,t)$ in (22.2).

function $u(x,t)$ is now piecewise defined by

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0, \\ x/t & \text{if } 0 < x \leq t, \\ 1 & \text{if } t < x. \end{cases} \tag{22.2}$$

The four frames of animation displayed in Figure 22.4 show that the profile of the solution "thins out" or "rarefies" as time increases. Such a function is an example of a *rarefaction wave.*

Note that although the function $u(x,t)$ defined in (22.2) is continuous for $t > 0$, the derivatives u_t and u_x do not exist along the lines $x = 0$ and $x = t$ and so u does not satisfy the differential equation $u_t + uu_x = 0$ at these points. This function, however, satisfies the conditions to be a *weak solution* of $u_t + uu_x = 0$, as we will describe later in Section 25.2.

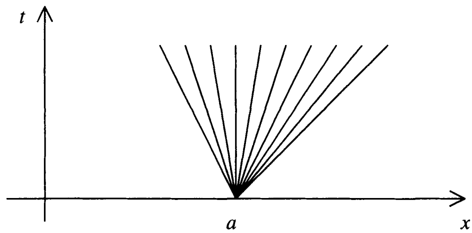


Figure 22.5. Characteristics for a rarefaction wave $u(x,t) = g((x-a)/t)$.

In general, a **rarefaction wave** is a nonconstant function of the form $u(x,t) = g((x-a)/t)$. The lines $(x-a)/t = c$ in the xt–plane are often called characteristics since u is constant along them; however, they are not constructed by the characteristic equation $dx/dt = c(u)$ derived from $u_t + c(u)u_x = 0$. These lines are distinguished by their fan shape originating from the point $x = a$ on the x–axis (Figure 22.5).

Exercise 22.2. Find a rarefaction wave solution of

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } x \le 0, \\ 2 & \text{if } x > 0. \end{cases}$$

22.2. Stopped traffic at a green light

Suppose traffic is backed up indefinitely in one direction behind a red light. The light, located at position $x = 0$, turns green at time $t = 0$ and the traffic begins to move forward. As shown in Figure 22.6, it will be assumed that prior to the changing of the light, traffic behind the light is at its maximum density u_1 and no traffic exists ahead of the light.

Using the constitutive equation $\phi = v_1(u - u^2/u_1)$ derived from the linear velocity model $v = v_1(1 - u/u_1)$, an initial value problem

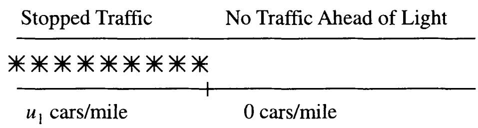


Figure 22.6. Traffic stopped at a red light.

which describes the traffic density u after the light turns green is

$$u_t + v_1(1 - 2u/u_1)u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$\text{(22.3)} \qquad u(x,0) = \begin{cases} u_1 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The characteristic lines for this initial value problem are of the form $x = c(u(x_0,0))t + x_0$ with c given by $c(u) = v_1(1 - 2u/u_1)$. Characteristics which start at points $(x_0, 0)$ on the negative $x-$axis ($x_0 < 0$) have speed

$$c(u(x_0,0)) = c(u_1) = v_1(1 - 2u_1/u_1) = -v_1,$$

while those starting at points on the positive $x-$axis have speed

$$c(u(x_0,0)) = c(0) = v_1(1 - 0) = v_1.$$

The resulting characteristic lines are then

$$x = -v_1 t + x_0 \quad \text{if } x_0 \le 0,$$
$$x = v_1 t + x_0 \quad \text{if } x_0 > 0.$$

The characteristic diagram shown in Figure 22.7 separates into three parts: $x < -v_1 t$, $-v_1 t < x < v_1 t$, and $x > v_1 t$. No characteristics enter the middle region; however, as shown in the following exercise, a rarefaction wave can be constructed to fill this wedge-shaped area. The resulting rarefaction wave solution is then

$$\text{(22.4)} \qquad u(x,t) = \begin{cases} u_1 & \text{if } x \le -v_1 t, \\ \frac{1}{2}u_1\left(1 - \frac{1}{v_1}\frac{x}{t}\right) & \text{if } -v_1 t < x < v_1 t, \\ 0 & \text{if } x \ge v_1 t. \end{cases}$$

Exercise 22.3. Construct the rarefaction wave solution (22.4) of the initial value problem (22.3) by using the method of characteristics in

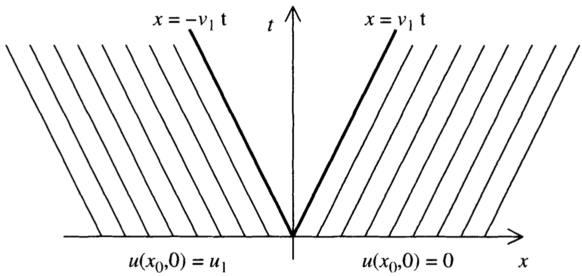


Figure 22.7. Characteristic lines of the initial value problem (22.3).

the regions $x < -v_1 t$ and $x > v_1 t$, and filling the middle wedge-shaped region $-v_1 t < x < v_1 t$ with a rarefaction wave.

Exercise 22.4. By picking values for u_1 and v_1, construct a specific example of the solution (22.4) and animate the result.

Exercise 22.5. In the xt–diagram of the function (22.4), what do the lines $x = -v_1 t$ and $x = v_1 t$ represent in terms of the traffic?

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Chapter 23

An Example with Rarefaction and Shock Waves

In general, nonlinear conservation laws may possess solutions which are constructed using a combination of shock and rarefaction waves. In this chapter we will construct an example of such a solution.

Consider the initial value problem for Burgers' equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } x \geq 1. \end{cases} \tag{23.1}$$

With $c(u) = u$, the characteristics $x = c(u(x_0, 0))t + x_0$ are

$$\begin{aligned} x &= 0 \cdot t + x_0 \quad \text{if } x_0 \leq 0, \\ x &= 1 \cdot t + x_0 \quad \text{if } 0 < x_0 < 1, \\ x &= 0 \cdot t + x_0 \quad \text{if } x_0 \geq 1. \end{aligned}$$

The characteristic diagram shown in Figure 23.1 has intersecting characteristics as well as a wedge-shaped region with no characteristics. Since u is constant along characteristics lines, the initial condition and the characteristic diagram show that $u(x,t) = 0$ for $x < 0$, $u(x,t) = 1$

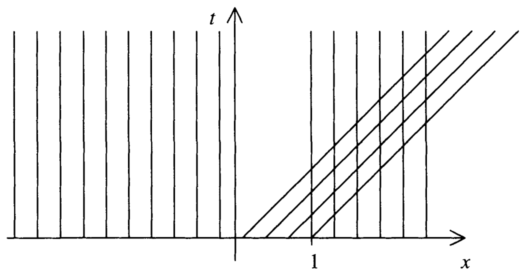


Figure 23.1. Characteristics of the initial value problem (23.1).

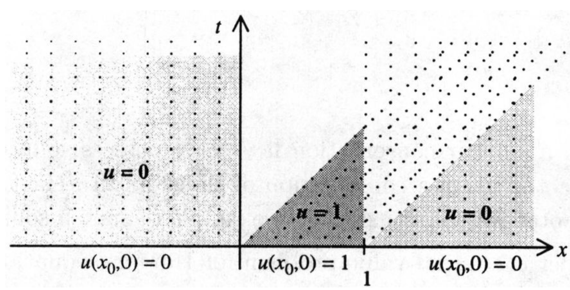


Figure 23.2. The value of u is constant along characteristics in regions of single characteristics.

for $0 < t < x < 1$, and $u(x,t) = 0$ for $0 < t < x - 1 < \infty$ (see Figure 23.2). A piecewise smooth solution to (23.1) will be completed using a combination of shock and rarefaction waves in the remaining regions of the xt−plane.

Step 1: A rarefaction. We will begin by constructing a rarefaction wave to fill the wedge-shaped region in the xt−plane that does not contain any characteristic lines. As shown in Section 22.1, a rarefaction wave solution of $u_t + uu_x = 0$ with a fan of characteristic lines originating from $(0,0)$ is

$$u(x,t) = \frac{x}{t}.$$

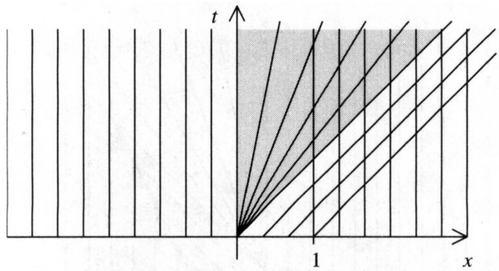


Figure 23.3. Characteristics of the rarefaction $u(x,t) = x/t$ fill the wedge-shaped region originating from the origin.

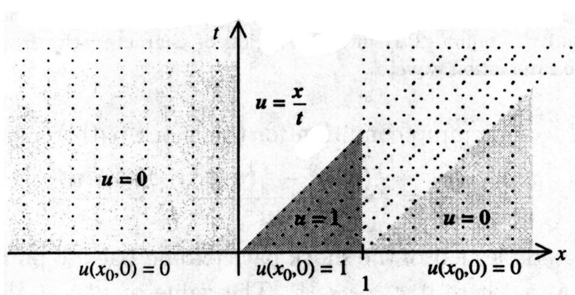


Figure 23.4. The xt–diagram of the solution including the rarefaction wave.

Drawing a fan of characteristic lines for this rarefaction in the triangular wedge results in the characteristic diagram shown in Figure 23.3 and the updated xt–diagram in Figure 23.4.

Exercise 23.1. The characteristic diagram in Figure 23.3 shows a region of intersecting characteristics near the x–axis. Sketch a possible shock path in Figure 23.3 starting at the point $(1, 0)$.

Step 2: A shock. The diagram in Figure 23.3 shows intersecting characteristics with a breaking time of $t_b = 0$. The next step will be to construct a shock path, starting at the point $(x,t) = (1,0)$, which separates the characteristics $x = t + x_0$ from the vertical lines $x = x_0$. With the flux $\phi(u) = \frac{1}{2}u^2$ from Burgers' equation $u_t + uu_x = 0$, the

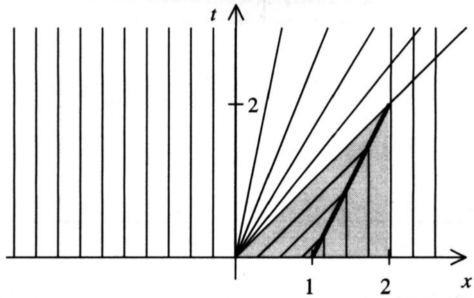


Figure 23.5. Shock path $x_s(t) = \frac{1}{2}t + 1$ for $0 \leq t \leq 2$ separates a region where $u(x,t) = 1$ from a region where $u(x,t) = 0$. The shock path will need to be extended beyond the point $(2,2)$ into the region of characteristics from the rarefaction wave.

Rankine-Hugoniot jump condition for the shock path becomes

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}.$$

The characteristics left of the shock path extend back to points $(x_0, 0)$ on the x−axis where $0 < x_0 < 1$. The value of $u(x,t)$ along these lines will be $u(x,t) = u(x_0, 0) = 1$, so the value of $u(x,t)$ as (x,t) approaches the shock path from the left is $u^- = 1$. Similarly, the characteristics to the right of the shock path are vertical lines which extend back to points $(x_0, 0)$ on the x−axis where $x_0 > 1$. The value of $u(x,t)$ along these lines will be $u(x,t) = u(x_0, 0) = 0$, so the value of $u(x,t)$ as (x,t) approaches the shock path from the right is $u^+ = 0$. The jump condition for the path then becomes

$$\frac{dx_s}{dt} = \frac{(1) + (0)}{2} = \frac{1}{2},$$

which gives $x_s = \frac{1}{2}t + k$. The constant k is found using the condition that the shock starts at $(x_s, t) = (1, 0)$. In this case $k = 1$, and the resulting shock path is

$$x_s = \tfrac{1}{2}t + 1, \quad 0 \leq t \leq 2.$$

As shown in Figure 23.5, this part of the shock path ends at $t = 2$, where the vertical characteristics begin intersecting the characteristics inserted for the rarefaction wave.

Step 3: Extension of the shock. The shock path constructed in Step 2 separates the characteristics $x = t + x_0$ from the vertical lines $x = x_0$. As a final step in the construction of $u(x,t)$, the shock will be extended from $(x,t) = (2,2)$ into the region $t > 2$ where the vertical lines $x = x_0$ intersect the fan of characteristics from the rarefaction wave (Figure 23.5).

As in Step 2, the jump condition for the shock path is

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{u^+ + u^-}{2}.$$

The characteristics to the right of the shock are vertical lines which extend back to points $(x_0, 0)$ on the $x-$axis with $x_0 > 1$. The value of $u(x,t)$ along these lines will be $u(x,t) = u(x_0, 0) = 0$, so the value of $u(x,t)$ as (x,t) approaches the shock path from the right is $u^+ = 0$. To the left of the path, we have already determined that the value of u is $u(x,t) = x/t$ from the rarefaction wave, so the value of $u(x,t)$ as (x,t) approaches the path from the left is $u^- = x/t$. The jump condition for points on the shock path is then

$$\frac{dx_s}{dt} = \frac{0 + x_s/t}{2} = \frac{x_s}{2t}.$$

This first order differential equation for x_s is separable; rewriting the equation as

$$\frac{1}{x_s}\frac{dx_s}{dt} = \frac{1}{2t}$$

and integrating shows that $\ln x_s = \ln\sqrt{t} + k$, and so $x_s = k_1\sqrt{t}$ for some constant k_1. Since this part of the shock path starts at the point $(x,t) = (2,2)$, the condition $x_s(2) = 2$ determines that $k_1 = \sqrt{2}$, and so the shock path here is

$$x_s = \sqrt{2t}, \quad t \geq 2.$$

As shown in Figure 23.6, this curve separates the region of rarefaction characteristics from the vertical characteristics for time $t \geq 2$.

The characteristic diagram in Figure 23.6 completes the construction of a piecewise smooth solution to the initial value problem (23.1); the final $xt-$diagram of the solution is shown in Figure 23.7. During

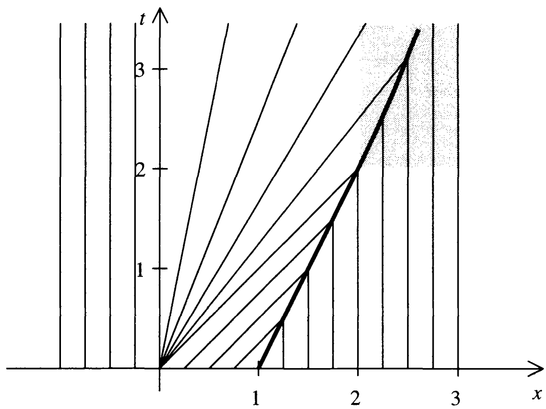


Figure 23.6. Extending the shock path by $x_s = \sqrt{2t}$ for $t \geq 2$ to separate the region of rarefaction characteristics from the vertical characteristics.

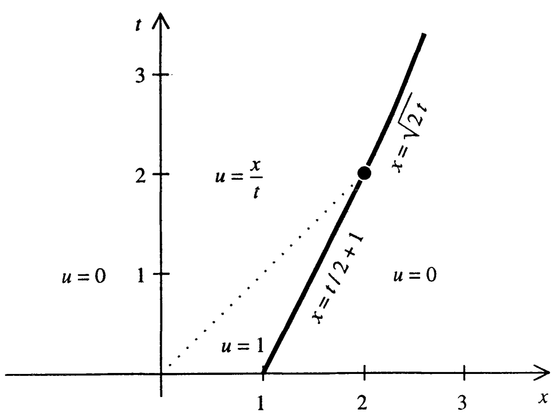


Figure 23.7. An xt–diagram for a function $u(x,t)$ consisting of a shock and a rarefaction.

the first two units of time, the profile of $u(x,t)$ is given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ x/t & \text{if } 0 < x < t, \\ 1 & \text{if } t < x < \frac{1}{2}t + 1, \\ 0 & \text{if } \frac{1}{2}t + 1 < x. \end{cases} \tag{23.2}$$

Once past time $t = 2$, the profile of $u(x,t)$ is defined by

$$(23.3) \qquad u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ x/t & \text{if } 0 < x < \sqrt{2t}, \\ 0 & \text{if } \sqrt{2t} < x. \end{cases}$$

Exercise 23.2. By taking slices of the xt–diagram shown in Figure 23.7, sketch the profile of $u(x,t)$ at times $t = 0, 0.5, 1, 1.5, 2, 2.5, 3$. In particular, notice (a) the formation and "thinning out" of a rarefaction wave starting at time $t = 0$, and (b) the movement of a shock discontinuity.

Exercise 23.3. Piecewise defined functions can be animated using the Heaviside function,

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

For example, during the first two units of time, the function $u(x,t)$ defined by (23.2) can be represented as

$$u(x,t) = (x/t)H(x)H(t-x) + H(x-t)H(t/2+1-x).$$

Using the Heaviside function, animate the function $u(x,t)$ defined in (23.2) and (23.3).

Exercise 23.4. Suppose that uniform traffic with density u_0 cars per mile approaches the end of a line of traffic stopped at a red light. Ahead of the light there are no cars, while the stopped traffic is at its maximum density u_1 cars per mile. At time $t = 0$, the red light turns green and the front of the line of stopped traffic begins to move forward. One model for the resulting traffic density is

$$(23.4) \qquad \begin{aligned} &u_t + v_1(1 - 2u/u_1)u_x = 0, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \begin{cases} u_0 & \text{if } x < -L, \\ u_1 & \text{if } -L < x < 0, \\ 0 & \text{if } x > 0. \end{cases} \end{aligned}$$

Assume that the incoming traffic has a density of $u_0 = \frac{1}{2}u_1$.

(a) Show that the characteristics for the initial value problem (23.4) are given by

$$\begin{aligned} x &= v_1 t + x_0 && \text{if } 0 < x_0, \\ x &= -v_1 t + x_0 && \text{if } -L < x_0 < 0, \\ x &= x_0 && \text{if } x_0 < -L. \end{aligned}$$

Sketch the resulting characteristic diagram.

(b) Find a solution $u(x, t)$ of the initial value problem (23.4) using a combination of shock and rarefaction waves.

(c) The companion MATLAB software (page xiii) includes a script **`wvtraf`** for animating the solution of this traffic flow problem. At the MATLAB prompt, type **`wvtraf`** and use the graphical interface to animate the solution from part (b).

Chapter 24

Nonunique Solutions and the Entropy Condition

Rarefaction and shock waves are special solutions of conservation laws that exhibit wave behavior. In the process of constructing them, however, we have relaxed the notion of "solution" from a function $u(x,t)$ which satisfies $u_t + \phi_x = 0$ for all (x,t), to a piecewise smooth solution which satisfies the integral form of the conservation law where u is not continuous. In this chapter, we will see that this more general notion of solution makes it possible for an initial value problem to possess many different solutions. The *entropy condition* will then be introduced as an example of a condition which is used to select one solution over all others.

24.1. Nonuniqueness of piecewise smooth solutions

The rarefaction wave from Section 22.1,

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0, \\ x/t & \text{if } 0 < x < t, \\ 1 & \text{if } x \geq t, \end{cases} \tag{24.1}$$

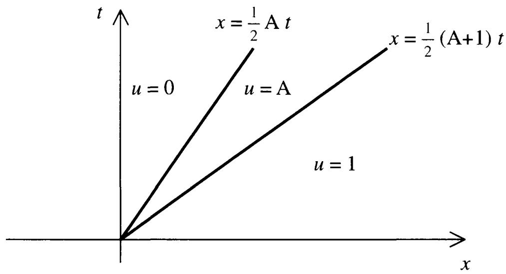


Figure 24.1. A shock wave solution of the initial value problem (24.2) with two shock paths.

was constructed as a piecewise smooth solution of the initial value problem

$$(24.2) \qquad \begin{aligned} &u_t + uu_x = 0, \quad -\infty < x < \infty,\ t > 0, \\ &u(x,0) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases} \end{aligned}$$

It is also possible, however, to find other solutions of this problem using shocks waves. In fact, if A is any number satisfying $0 < A < 1$, then the function

$$(24.3) \qquad u(x,t) = \begin{cases} 0 & \text{if } x \le \frac{1}{2}At, \\ A & \text{if } \frac{1}{2}At < x < \frac{1}{2}(A+1)t, \\ 1 & \text{if } \frac{1}{2}(A+1)t \le x, \end{cases}$$

represented by the xt–diagram in Figure 24.1 is a shock wave solution with two shock paths (see Exercise 24.1). Thus there are many solutions of the initial value problem (24.2)—a rarefaction wave solution and an infinite number of shock wave solutions.

Exercise 24.1. Consider the function $u(x,t)$ given by (24.3).

(a) Verify that $u(x,t)$ satisfies $u_t + uu_x = 0$ in each of the three regions $x < \frac{1}{2}At$, $\frac{1}{2}At < x < \frac{1}{2}(A+1)t$, and $x > \frac{1}{2}(A+1)t$.

(b) Verify that the paths of discontinuity $x_s = \frac{1}{2}At$ and $x_s = \frac{1}{2}(A+1)t$ satisfy the Rankine-Hugoniot jump condition.

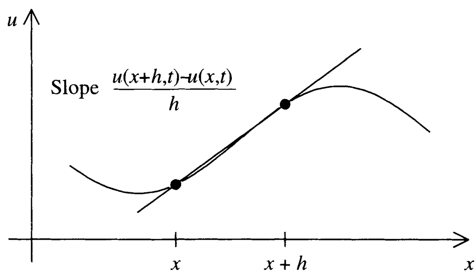
24.2. The entropy condition

When an initial value problem has more than one solution, additional information must be specified if one particular solution is to be selected. In gas dynamics, for example, the *entropy condition* is used to select a solution which is most physically realistic.

A function $u(x,t)$ satisfies the **entropy condition** if it is possible to find a positive constant E so that

$$\frac{u(x+h,t)-u(x,t)}{h} \le \frac{E}{t}$$

for all $t > 0$, $h > 0$, and x. Graphically, this is a condition on the slope of the profile of $u(x,t)$ at each time t—the slope between any two points on the profile (secant slope) at time t is less than E/t:



Note that this condition restricts how large the *positive* secant slope can be, and does not prohibit the curve from having steep negative slopes. Furthermore, the bound E/t restricting the size of positive slopes decreases to zero as t increases.

For the initial value problem (24.2) in the previous section, there are an infinite number of shock wave solutions given by (24.3). Figure 24.2 shows the profile of these solutions, and indicates that large positive secant slopes are possible by picking x and $x+h$ on opposite sides of the shock. The secant slope

$$\frac{u(x+h,t)-u(x,t)}{h} = \frac{1-A}{h}$$

grows arbitrarily large as x and $x+h$ approach the location of the jump, so it is not possible to find a constant E such that this secant

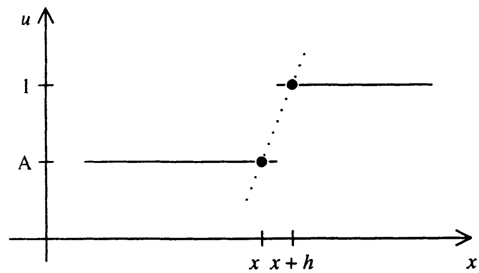


Figure 24.2. Large positive secant slopes occur in the profiles of the shock wave solutions (24.3).

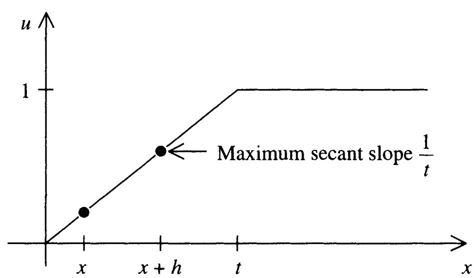


Figure 24.3. Maximum positive secant slope is $1/t$ in the profiles of the rarefaction wave (24.1).

slope is less than E/t for all x and $h > 0$. The shock wave solutions (24.3) do not satisfy the entropy condition.

The rarefaction wave (24.1), however, does satisfy the entropy condition. The profile of this function at time t shown in Figure 24.3 indicates that a maximum positive secant slope of $1/t$ occurs when x and $x+h$ are between 0 and t. For this function the entropy condition is met by picking $E = 1$, since

$$\frac{u(x+h,t) - u(x,t)}{h} \leq \frac{1}{t}.$$

The entropy condition would then select this rarefaction wave solution over the shock waves solutions in the initial value problem (24.2).

The entropy condition plays an important role in the design of numerical methods for constructing approximations to solutions of conservation laws. Since a conservation law may possess several solutions, care must be taken to ensure that the numerical method not only converges, but converges to the desired solution. For further reading on the entropy condition, its variations, and its role in numerical algorithms, see either [**LeV**] or [**Smo**].

Exercise 24.2. Find a rarefaction wave solution for

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } x \le 0, \\ 2 & \text{if } x > 0, \end{cases}$$

and sketch a profile of $u(x,t)$ at time t. What is the maximum secant slope of this profile? Does this rarefaction wave solution satisfy the entropy condition?

Exercise 24.3. Find a rarefaction wave solution for

$$u_t + u^2 u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Does this rarefaction wave solution satisfy the entropy condition?

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Chapter 25

Weak Solutions of Conservation Laws

25.1. Classical solutions

Constructing solutions of conservation laws by piecing together shocks and rarefactions can become quite tedious if the initial condition is anything more than a very simple function. Furthermore, constructing a particular solution is sometimes not as important as determining more general properties of the conservation law. In this chapter the *weak form* of a conservation law is introduced as an alternative to the differential equation form $u_t + \phi_x = 0$. This view of the conservation law has several mathematical advantages over the differential equation form.

The solutions of differential equations that we have focused on are often called *classical solutions* in order to distinguish them from the *weak solutions* described in the next section. Consider the general initial value problem

$$u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x, 0) = u_0(x),$$

where $\phi(x, t)$ has continuous first derivatives and $u_0(x)$ is continuous. A function $u(x, t)$ is called a **classical solution** of this initial value problem if (a) u is continuous for all x and $t \geq 0$, (b) u_x and u_t exist and are continuous for all x and $t > 0$, (c) u satisfies $u_t + \phi_x = 0$ for all x and $t > 0$, and (d) $u(x,0) = u_0(x)$ for all x.

The notion of weak solution will allow us to proceed directly to functions $u(x,t)$ which are not necessarily continuous or differentiable, but are solutions in a different sense.

25.2. The weak form of a conservation law

The *weak form* of $u_t + \phi_x = 0$ is an alternative integral form of the conservation law. The underlying idea is to use special functions of x and t, called *test functions*, to examine the solution of $u_t + \phi_x = 0$ in regions of the xt–plane. A real valued function $T(x,t)$ is called a **test function** if

(a) T_t and T_x exist and are continuous for all (x,t), and

(b) there is some circle in the xt–plane such that $T(x,t) = 0$ for all (x,t) on or outside the circle.

An example of a test function is

$$T(x,t) = \begin{cases} \exp\left(\frac{-1}{1-x^2-t^2}\right) & \text{if } x^2+t^2 < 1, \\ 0 & \text{if } x^2+t^2 \geq 1, \end{cases}$$

whose graph is shown in Figure 25.1. The exponential decay of $T(x,t)$ to zero as (x,t) approaches the boundary of the circle $x^2+t^2 = 1$ from the inside leads to this function having continuous first derivatives T_t and T_x for all (x,t), even on the unit circle.

To derive the weak form of a conservation law, begin by assuming that u is a classical solution of

$$u_t + \phi_x = 0, \quad -\infty < x < \infty,\ t > 0, \tag{25.1a}$$

$$u(x,0) = u_0(x), \tag{25.1b}$$

and let $T(x,t)$ be any test function. The product $T(x,t)u(x,t)$ is now a function which is zero at every point (x,t) on and outside a circle in the xt–plane, so this product isolates a portion of u.

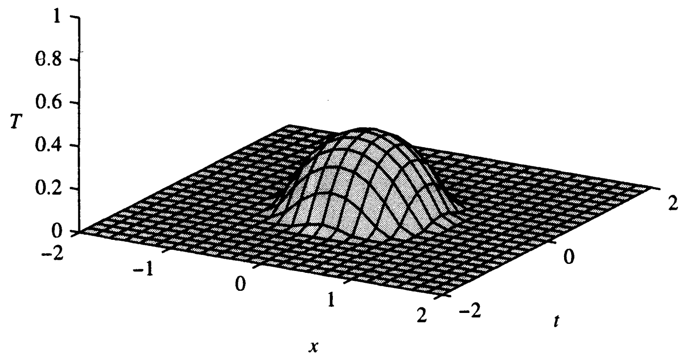

Figure 25.1. A test function.

Multiplying the differential equation (25.1a) by $T(x,t)$ and integrating over all possible x and all $t \geq 0$ gives

$$\int_0^\infty \int_{-\infty}^\infty \left[u_t(x,t)T(x,t) + \phi_x(x,t)T(x,t)\right]dxdt = 0. \tag{25.2}$$

The left side can be written as the sum of two integrals I_1 and I_2, where

$$I_1 = \int_0^\infty \int_{-\infty}^\infty u_t(x,t)T(x,t)dxdt,$$
$$I_2 = \int_0^\infty \int_{-\infty}^\infty \phi_x(x,t)T(x,t)dxdt.$$

Interchanging the order of integration in the double integral I_1 and applying integration by parts to the resulting inside integral rewrites I_1 as

$$\begin{aligned} I_1 &= \int_{-\infty}^\infty \left[\int_0^\infty u_t(x,t)T(x,t)dt\right] dx \\ &= \int_{-\infty}^\infty \left[u(x,t)T(x,t)\Big|_{t=0}^{t\to\infty} - \int_0^\infty u(x,t)T_t(x,t)dt\right] dx. \end{aligned}$$

The value of $u(x,t)T(x,t)$ is zero as $t \to \infty$ since $T(x,t)$ is zero for all (x,t) outside some circle in the xt-plane. The value of $u(x,0)T(x,0)$

is $u_0(x)T(x,0)$ by the initial condition (25.1b). The expression for I_1 is then

$$(25.3)\quad I_1 = -\int_{-\infty}^{\infty} u_0(x)T(x,0)dx - \int_0^{\infty}\int_{-\infty}^{\infty} u(x,t)T_t(x,t)dxdt.$$

A similar calculation can be carried out for I_2. Applying integration by parts to the inside integral of the double integral I_2 results in

$$\begin{aligned} I_2 &= \int_0^{\infty}\left[\int_{-\infty}^{\infty} \phi_x(x,t)T(x,t)dx\right]dt \\ &= \int_0^{\infty}\left[\phi(x,t)T(x,t)\Big|_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} \phi(x,t)T_x(x,t)dx\right]dt. \end{aligned}$$

The value of $u(x,t)T(x,t)$ is zero as $x \to \pm\infty$ since $T(x,t)$ is zero for all (x,t) outside some circle in the xt–plane, so

$$(25.4)\qquad I_2 = -\int_0^{\infty}\int_{-\infty}^{\infty} \phi(x,t)T_x(x,t)dxdt.$$

Using the two calculations (25.3) and (25.4) for I_1 and I_2, the integral of the conservation law $u_t + \phi_x = 0$ in (25.2) can be rewritten as

$$(25.5)\quad \boxed{\begin{aligned} &\int_0^{\infty}\int_{-\infty}^{\infty} \big(u(x,t)T_t(x,t) + \phi(x,t)T_x(x,t)\big)dxdt \\ &\qquad\qquad + \int_{-\infty}^{\infty} u_0(x)T(x,0)dx = 0. \end{aligned}}$$

This is called the **weak form** of the initial value problem (25.1) for the conservation law $u_t + \phi_x = 0$.

Note that the weak form (25.5) does not involve any derivatives of $u(x,t)$. A **weak solution** of the initial value problem (25.1) is a function $u(x,t)$ which satisfies (25.5) for every test function $T(x,t)$. For a weak solution there is no requirement that u_t or u_x even exist. Furthermore, the partial differential equation and the initial condition in (25.1) are both accounted for in this single equation.

Example 25.1. Consider the initial value problem

$$\begin{aligned} u_t + u^2u_x &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \frac{1}{1+x^2}. \end{aligned}$$

The flux for this conservation law is $\phi(u) = u^3/3$. Taking this flux and the initial function $u_0(x) = 1/(1+x^2)$ in (25.5) gives the weak form of the initial value problem as

$$\int_0^\infty \int_{-\infty}^\infty \left(u(x,t)T_t(x,t) + \tfrac{1}{3}u^3(x,t)T_x(x,t)\right) dxdt + \int_{-\infty}^\infty \frac{T(x,0)}{1+x^2}dx = 0$$

for all test functions $T(x,t)$.

Exercise 25.2. Find the weak form of the following initial value problems:

(a) $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

(b) $u_t + (e^u)_x = 0$, $u(x,0) = \begin{cases} 2 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$

(c) $u_t + v_{max}(1 - 2u/u_{max})u_x = 0$, $u(x,0) = \begin{cases} u_{max} & \text{if } x \le 0, \\ u_0 & \text{if } x > 0. \end{cases}$

Exercise 25.3. By following the derivation of (25.5), show that the weak form of $u_t + \phi_x = f$ is

$$\int_0^\infty \int_{-\infty}^\infty \left(u(x,t)T_t(x,t) + \phi(x,t)T_x(x,t) + f(x,t)T(x,t)\right)dxdt + \int_{-\infty}^\infty u_0(x)T(x,0)dx = 0.$$

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An Introduction to the Mathematical Theory of Waves

Roger Knobel

Linear and nonlinear waves are a central part of the theory of PDEs. This book begins with a description of one-dimensional waves and their visualization through computer-aided techniques. Next, traveling waves are covered, such as solitary waves for the Klein-Gordon and KdV equations. Finally, the author gives a lucid discussion of waves arising from conservation laws, including shock and rarefaction waves. As an application, interesting models of traffic flow are used to illustrate conservation laws and wave phenomena.

This book is based on a course given by the author at the IAS/Park City Mathematics Institute. It is suitable for independent study by undergraduate students in mathematics, engineering, and science programs.

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